A CLASS OF INTEGRABLE SPIN CALOGERO-MOSER SYSTEMS II: EXACT SOLVABILITY

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Dedicated with respect and admiration to Percy Deift on the occasion of his 60th birthday

Abstract. In a previous paper, we introduce a class of integrable spin Calogero-Moser systems associated with the classical dynamical r-matrices with spectral parameter. Here the main purpose is to give explicit solutions of several factorization problems associated with infinite dimensional Lie groupoids which will allow us to write down the solutions of these integrable models.

1. Introduction.

In [LX1],[LX2], we introduce a class of spin Calogero-Moser (CM) systems associated with so-called classical dynamical r-matrices with spectral parameter, as defined and classified in [EV] for simple Lie algebras, following the pioneering work of Felder [F] and Felder and Wieczerkowski [FW] in which the classical dynamical Yang-Baxter equation (CDYBE) with spectral parameter was introduced and studied in the context of conformal field theory. The main purpose of this sequel is to show how to obtain the explicit solutions of the associated integrable systems in [LX2] by using the factorization method developed in [L2].

The spin CM systems constructed in the afore-mentioned papers are of three types-rational, trigonometric and elliptic. Indeed, for each of the canonical forms of the three types of z-dependent classical dynamical r-matrices in [EV], there is an intrinsic way to construct an associated spin CM system and its realization in the dual bundle of a corresponding coboundary dynamical Lie algebroid. In this way, we are led to a family of rational spin CM systems parametrized by subsets $\Delta' \subset \Delta$ which are closed with respect to addition and multiplication by -1. Here Δ is the root system associated with a complex simple Lie algebra $\mathfrak{g}$ with a fixed Cartan subalgebra $\mathfrak{h}$. In the trigonometric case, we also have a family but here the family is parametrized by subsets π' of a fixed simple system $\pi \subset \Delta$. Finally, we have an elliptic spin Calogero-Moser system for every simple Lie algebra. Let us summarize

a few key features of these Hamiltonian systems and their realization spaces as follows: (a) in each case the phase space P is a Hamiltonian H-space (with equivariant momentum map J) which admits an H-equivariant realization in the dual bundle $A\Gamma$ of an infinite dimensional coboundary dynamical Lie algebroid $A^*\Gamma$ and the Hamiltonian is the pullback of a natural invariant function on $A\Gamma$ under the realization map ρ, (b) the coboundary dynamical Lie algebroids involved are associated with solutions of the modified dynamical Yang-Baxter equation (mDYBE), (c) the pullbacks of the natural invariant functions on $A\Gamma$ by ρ do not Poisson commute everywhere on P, but they do so on the fiber $J^{-1}(0)$ in all cases, (d) the reduced Hamiltonian system on $J^{-1}(0)/H$ admits a natural collection of Poisson commuting integrals. As a matter of fact, we now know from [L2] that $A\Gamma$ is also a Hamiltonian H-space and it follows from the same work that the integrable flows on the reduced space are realized on the Poisson quotient $\gamma^{-1}(0)/H$, where $\gamma : A\Gamma \longrightarrow \mathfrak{h}^*,\ (q, \lambda, X) \mapsto \lambda$ is the momentum map of the H-action on $A\Gamma$. Indeed we have $\rho(J^{-1}(0)) \subset \gamma^{-1}(0)$ in each case. Since ρ is H-equivariant, it therefore induces a Poisson map between the corresponding Poisson quotients.

We now turn to our approach on exact solvability. As we showed in [L2], the mDYBE is associated with factorization problems on trivial Lie groupoids the solutions of which provide an effective method to integrate the generalized Lax equations on $\gamma^{-1}(0)$. (See [L1] for the groupoid version.) In the context of our spin CM systems, the factors which appear in the factorization problems are elements of certain Lie subgroupoids of trivial Lie groupoids of the form $\Gamma = U \times LG \times U$, where LG is the loop group associated with a simple Lie group G, and U is an open subset of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g} = Lie(G)$. Consequently, it is essential to have a precise description of the elements which belong to these Lie subgroupoids. As a first step towards the solutions of the factorization problems, we begin by analyzing the corresponding decompositions on the infinitesimal level, i.e., at the level of Lie algebroids. Here, the r-matrix $\mathcal{R} : A^*\Gamma \longrightarrow A\Gamma$ of the coboundary dynamical Lie algebroid $A^*\Gamma$ plays an important role. Indeed, for associated bundle maps $\mathcal{R}^{\pm} : A^*\Gamma \longrightarrow A\Gamma$, $Im\,\mathcal{R}^{\pm}$ are Lie subalgebroids of the trivial Lie algebroid $A\Gamma$ and the infinitesimal version of factorization is the decomposition

$$(0_q, X, 0) = \frac{1}{2}\mathcal{R}^+(0_q, X, 0) - \frac{1}{2}\mathcal{R}^-(0_q, X, 0)$$

where the element $(0_q, X, 0)$ on the left hand side of the above expression is in the adjoint bundle of $A\Gamma$. Thus it is essential to be able to describe the elements of $Im\,\mathcal{R}^{\pm}$. As it turns out, $Im\,\mathcal{R}^+ = \bigcup_{q\in U}\{0_q\} \times L^+\mathfrak{g} \times \mathfrak{h}$ in all three cases, where

$L^+\mathfrak{g}$ is the Lie subalgebra of the loop algebra $L\mathfrak{g}$ consisting of convergent power series $\sum_0^\infty X_n z^n$. On the other hand, $Im\,\mathcal{R}^-$ in each of the three cases is given by a matched product $\mathcal{I}^- \bowtie \mathcal{Q}$ (in the sense of Mokri [Mok]) where the ideal $\mathcal{I}^-$ coincides with the adjoint bundle of $Im\,\mathcal{R}^-$ and $\mathcal{Q}$ is a Lie subalgebroid of $Im\,\mathcal{R}^-$ isomorphic to TU. In spite of this, the method of solution of the factorization problems is quite different in the three cases under consideration. That this is so is due to the difference in the analyticity properties of the elements in the ideals $\mathcal{I}^-$. In the rational case and trigonometric case, we can reduce the factorization problems to finite dimensional problems due to some special features of the Lax operators. (Of course, this is also a reflection of the underlying character of the flows.) However, this is not so in the elliptic case-here we will only do things for the classical Lie algebras and indeed we will only give details for $\mathfrak{g} = sl(N,\mathbb{C})$ as the arguments for the other classical Lie algebras are similar. In this case, the explicit solution of the factorization problem is obtained with the help of multi-point Baker-Akheizer functions connected with the spectral curve C. Thus the solution of the equations can be expressed in terms of Riemann theta functions.

The paper is organized as follows. In Section 2, we present a number of basic results which will be used throughout the paper. More specifically, we will begin with a review on the geometric scheme to construct integrable systems based on realization in the dual bundles of coboundary dynamical Lie algebroids and the factorization theory which we mentioned earlier. Then we will turn our attention to a subclass of coboundary dynamical Lie algebroids defined by the classical dynamical r-matrices with spectral parameter. We will also recall what we mean by spin Calogero-Moser systems associated with this subclass of coboundary dynamical Lie algebroids. In Section 3, we discuss the solution of the integrable rational spin Calogero-Moser systems by solving the corresponding factorization problem. In Section 4, we handle the trigonometric case. Finally in Section 5, we analyze the elliptic case.

To close, we would like to point out what was done in the paper [KBBT] so that the reader can better understand why a different method is required for our more general class of systems here. To cut the story short, what the authors considered in [KBBT] are the $gl(N,\mathbb{C})$-rational spin Calogero-Moser system of Gibbons and Hermsen [GH], as well as their trigonometric and elliptic counterparts. From our point of view, these are special cases which can be obtained from more general $gl(N,\mathbb{C})$-systems (see [BAB] and [ABB]) by restricting the matrix of 'spin variables' to some *special* coadjoint orbits of $gl(N,\mathbb{C})^* \simeq gl(N,\mathbb{C})$ which can be parametrized

by vectors $a_j, b_j \in \mathbb{C}^l$, $l < N$, $j = 1, \cdots, N$. The method for solving such systems in [KBBT] is based on the connection with the matrix KP equation and is specific to these special coadjoint orbits of $gl(N, \mathbb{C})^* \simeq gl(N, \mathbb{C})$. For a sketch of this method in the elliptic case together with an explanation of its limitations, we refer the reader to Remark 5.2.8 (a).

2. Invariant Hamiltonian systems associated with coboundary dynamical Lie algebroids and the factorization method.

In the first two subsections, we shall present a number of basic results from [L2] which will be used throughout the paper. In particular, we shall give a summary of the factorization theory. In the last subsection, we shall turn our attention to a subclass of coboundary dynamical Lie algebroids defined by the classical dynamical r-matrices with spectral parameter [LX2]. Since the paper is concerned with the solution of the class of integrable spin Calogero-Moser systems introduced in [LX2], we will recall its construction and its relation to this subclass of Lie algebroids in this last subsection.

2.1 Geometry of the modified dynamical Yang-Baxter equation.

Let G be a connected Lie group, $H \subset G$ a connected Lie subgroup, and $\mathfrak{g}$, $\mathfrak{h}$ their Lie algebras. We shall denote by $\iota : \mathfrak{h} \longrightarrow \mathfrak{g}$ the Lie inclusion. In what follows, the Lie groups and Lie algebras can be real or complex unless we specify otherwise.

If $U \subset \mathfrak{h}^*$ is a connected Ad^*_H-invariant open subset, we say that a smooth (resp. holomorphic) map $R : U \longrightarrow L(\mathfrak{g}^*, \mathfrak{g})$ (here and henceforth we denote by $L(\mathfrak{g}^*, \mathfrak{g})$ the set of linear maps from $\mathfrak{g}^*$ to $\mathfrak{g}$) is a classical dynamical r-matrix [EV] associated with the pair $(\mathfrak{g}, \mathfrak{h})$ iff R is pointwise skew symmetric

$$< R(q)(A), B >= - < A, R(q)B > \tag{2.1.1}$$

and satisfies the classical dynamical Yang-Baxter condition

$$\begin{aligned} &[R(q)A, R(q)B] + R(q)(ad^*_{R(q)A}B - ad^*_{R(q)B}A) \\ &+ dR(q)\iota^* A(B) - dR(q)\iota^* B(A) + d < R(A), B > (q) = \chi(A, B), \end{aligned} \tag{2.1.2}$$

for all $q \in U$, and all $A, B \in \mathfrak{g}^*$, where $\chi : \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}$ is G-equivariant. The dynamical r-matrix is said to be H-equivariant if and only if

$$R(Ad^*_{h^{-1}}q) = Ad_h \circ R(q) \circ Ad^*_h \tag{2.1.3}$$

for all $h \in H, q \in U$.

We shall equip $\Gamma = U \times G \times U$ with the trivial Lie groupoid structure over U with structure maps (target, source, ... , multiplication)

$$\begin{aligned} &\alpha(u,g,v) = u,\ \beta(u,g,v) = v,\ \epsilon(u) = (u,1,u),\ i(u,g,v) = (u,g^{-1},v) \\ &\qquad m((u,g,v),(v,g',w)) = (u,gg',w) \end{aligned} \tag{2.1.4}$$

and let $A\Gamma = Ker\, T\alpha|_{\epsilon(U)} = \bigcup_{q\in U}\{0_q\} \times \mathfrak{g} \times \mathfrak{h}^* \simeq TU \times \mathfrak{g}$ be its Lie algebroid with anchor map denoted by a. (See [CdSW] and [M1] for details.) Recall that associated with an H-equivariant classical dynamical r-matrix R there is a natural Lie algebroid bracket $[\cdot,\cdot]_{A^*\Gamma}$ on the dual bundle $A^*\Gamma$ of $A\Gamma$ [BKS],[L2] such that the pair $(A\Gamma, A^*\Gamma)$ is a Lie bialgebroid in the sense of MacKenzie and Xu [MX].(Lie bialgebroids are infinitesimal versions of the Poisson groupoids of Weinstein [W].) Throughout the paper, the pair $(A^*\Gamma, [\cdot,\cdot]_{A^*\Gamma})$ together with the anchor map a_* : $A^*\Gamma \longrightarrow TU$ given by

$$a_*(0_q, A, Z) = (q, \iota^* A - ad^*_Z q) \tag{2.1.5}$$

will be called the *coboundary dynamical Lie algebroid* associated to R.

A special case of (2.1.2) is the modified dynamical Yang-Baxter equation (mDYBE):

$$\begin{aligned} &[R(q)A, R(q)B] + R(q)(ad^*_{R(q)A}B - ad^*_{R(q)B}A) \\ &+ dR(q)\iota^*A(B) - dR(q)\iota^*B(A) + d < R(A), B > (q) \\ &= -\,[K(A), K(B)] \end{aligned} \tag{2.1.6}$$

where $K \in L(\mathfrak{g}^*, \mathfrak{g})$ is a nonzero symmetric map which satisfies $ad_X \circ K + K \circ ad^*_X = 0$ for all $X \in \mathfrak{g}$,i.e., K is G-equivariant. In [L2], the class of coboundary dynamical Lie algebroids associated with mDYBE was singled out and was shown to have some rather remarkable properties. We will restrict to this class of $(A^*\Gamma, [\cdot,\cdot]_{A^*\Gamma}, a_*)$ in the rest of the subsection.

Following [L1] and [L2], we introduce the bundle map

$$\mathcal{R} : A^*\Gamma \longrightarrow A\Gamma, (0_q, A, Z) \mapsto (0_q, -\iota Z + R(q)A, \iota^*A - ad^*_Z q) \tag{2.1.7}$$

and call it the *r-matrix of the Lie algebroid* $A^*\Gamma$. Also, define

$$\mathcal{K} : A^*\Gamma \longrightarrow A\Gamma, (0_q, A, Z) \mapsto (0_q, K(A), 0), \tag{2.1.8}$$

and set

$$\mathcal{R}^{\pm} = \mathcal{R} \pm \mathcal{K},\ R^{\pm}(q) = R(q) \pm K. \tag{2.1.9}$$

Proposition 2.1.1. *(a) $\mathcal{R}^{\pm}$ are morphisms of transitive Lie algebroids and, as morphisms of vector bundles over U, are of locally constant rank.*

(b) $Im\mathcal{R}^{\pm}$ are transitive Lie subalgebroids of $A\Gamma$.

In the rest of the subsection,we shall assume $\mathfrak{g}$ has a nondegenerate invariant pairing $(\cdot,\cdot)$ such that $(\cdot,\cdot)\mid_{\mathfrak{h}\times\mathfrak{h}}$ is also nondengenerate. Without loss of generality, we shall take the map $K:\mathfrak{g}^*\longrightarrow\mathfrak{g}$ in the above discussion to be the identification map induced by $(\cdot,\cdot)$. Indeed, with the identifications $\mathfrak{g}^*\simeq\mathfrak{g}$, $\mathfrak{h}^*\simeq\mathfrak{h}$, we have $K=id_{\mathfrak{g}}$. We shall regard $R(q)$ as taking values in $End(\mathfrak{g})$, and the derivatives as well as the dual maps are computed using $(\cdot,\cdot)$. Also, we have $ad^*\simeq -ad$, $\iota^*\simeq\Pi_{\mathfrak{h}}$, where $\Pi_{\mathfrak{h}}$ is the projection map to $\mathfrak{h}$ relative to the direct sum decomposition $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{h}^{\perp}$. We shall keep, however, the notation $A^*\Gamma$ although as a set it can be identified with $A\Gamma$.

We now introduce the following subbundles of the adjoint bundle $Ker\,a = \big\{(0_q,X,0)\mid q\in U, X\in\mathfrak{g}\big\}$ of $A\Gamma$:

$$\mathcal{I}^{\pm}=\big\{(0_q,X,0)\in Ker\,a\mid q\in U,\mathcal{R}^{\mp}(0_q,X,Z)=0\ \text{ for some }\ Z\in\mathfrak{h}\big\}. \tag{2.1.10}$$

Proposition 2.1.2. *(a) $\mathcal{I}^{\pm}$ are ideals of the transitive Lie algebroids $Im\mathcal{R}^{\pm}$.*

(b) Equip $Im\mathcal{R}^+/\mathcal{I}^+$ and $Im\mathcal{R}^-/\mathcal{I}^-$ with the quotient transitive Lie algebroid structures, then the map $\theta: Im\mathcal{R}^+/\mathcal{I}^+\longrightarrow Im\mathcal{R}^-/\mathcal{I}^-$ defined by

$$\theta(\mathcal{R}^+(0_q,X,Z)+\mathcal{I}^+_q)=\mathcal{R}^-(0_q,X,Z)+\mathcal{I}^-_q$$

is an isomorphism of transitive Lie algebroids.

Theorem 2.1.3. *(a) Every element $(0_q,X,0)\in Ker\,a$ admits a unique decomposition*

$$(0_q,X,0)=\mathcal{X}_+-\mathcal{X}_-$$

where $(\mathcal{X}_+,\mathcal{X}_-)\in(\mathcal{R}^+,\mathcal{R}^-)(\{0_q\}\times\mathfrak{g}\times\{0\})$ with $\theta(\mathcal{X}_++\mathcal{I}^+_q)=\mathcal{X}_-+\mathcal{I}^-_q$.

(b) The coboundary dynamical Lie algebroid $A^*\Gamma$ is isomorphic to the Lie subalgebroid

$$\Big\{(\mathcal{X}_+,\mathcal{X}_-)\in(Im\mathcal{R}^+\bigoplus_{TU}Im\mathcal{R}^-)_q\mid q\in U,\,\theta(\mathcal{X}_++\mathcal{I}^+_q)=\mathcal{X}_-+\mathcal{I}^-_q\Big\}$$

of $Im\mathcal{R}^+\bigoplus_{TU}Im\mathcal{R}^-$.

2.2 Invariant Hamiltonian systems and the factorization method.

We shall continue to use the same assumptions on $\mathfrak{g}$, K and R as in the latter part of Section 2.1. Although it is not entirely necessary to assume that R satisfies the mDYBE for Theorem 2.2.1, however, it serves our purpose here.

Let P be a Poisson manifold and suppose $\rho = (m, \tau, L) : P \longrightarrow U \times \mathfrak{h} \times \mathfrak{g} \simeq A\Gamma$ is a realization of P in the dual bundle $A\Gamma$ of the coboundary dynamical Lie algebroid $A^*\Gamma$, equipped with the Lie-Poisson structure, i.e., ρ is a Poisson map. Recall that $A\Gamma$ is a Hamiltonian H-space under the natural action $h.(q, \lambda, X) = (Ad^*_{h^{-1}} q, Ad^*_{h^{-1}} \lambda, Ad_h X)$ and the projection map $\gamma : A\Gamma \longrightarrow \mathfrak{h}^*$, $(q, \lambda, X) \mapsto \lambda$ is an equivariant momentum map. We assume:

A1. P is a Hamiltonian H-space with an equivariant momentum map $J : P \longrightarrow \mathfrak{h}$,
A2. the realization map ρ is H-equivariant,
A3. for some regular value $\mu \in \mathfrak{h}$ of J,

$$\rho(J^{-1}(\mu)) \subset \gamma^{-1}(0) = U \times \{0\} \times \mathfrak{g}. \tag{2.2.1}$$

Let $I(\mathfrak{g})$ be the ring of smooth ad-invariant functions on $\mathfrak{g}$, $i_\mu : J^{-1}(\mu) \longrightarrow P$ the inclusion map, and $\pi_\mu : J^{-1}(\mu) \longrightarrow J^{-1}(\mu)/H_\mu$ the canonical projection, where H_μ is the isotropy subgroup of μ for the H-action on P. Also, let Pr_3 denote the projection map onto the third factor of $U \times \mathfrak{h} \times \mathfrak{g} \simeq A\Gamma$. We consider H-invariant Hamiltonian systems on P, generated by Hamiltonians of the form $\mathcal{F} = L^* f$, where $f \in I(\mathfrak{g})$.

Theorem 2.2.1. *Under the above assumptions,*

(a) The Hamiltonian $\mathcal{F}$ descends to a unique function $\mathcal{F}_\mu$ on the reduced Poisson variety $P_\mu = J^{-1}(\mu)/H_\mu$. Moreover, $\mathcal{F}_\mu$ admits a natural family of Poisson commuting functions given by the reduction of functions in $L^* I(\mathfrak{g})$,

(b) If ψ_t is the induced flow on $\gamma^{-1}(0)$ generated by the Hamiltonian $Pr_3^* f$, and ϕ_t is the Hamiltonian flow of $\mathcal{F}$ on P, then under the flow ϕ_t, we have

$$\begin{aligned}
&\frac{d}{dt} m(\phi_t) = \Pi_{\mathfrak{h}} df(L(\phi_t)), \\
&\frac{d}{dt} \tau(\phi_t) = 0, \\
&\frac{d}{dt} L(\phi_t) = [\, L(\phi_t), R(m(\phi_t)) df(L(\phi_t)) \,] + dR(m(\phi_t))(\tau(\phi_t)) df(L(\phi_t))
\end{aligned}$$

where the term involving dR drops out on $J^{-1}(\mu)$. Moreover, the reduction ϕ_t^{red} of $\phi_t \circ i_\mu$ on P_μ defined by $\phi_t^{red} \circ \pi_\mu = \pi_\mu \circ \phi_t \circ i_\mu$ is a Hamiltonian flow of $\mathcal{F}_\mu$.

This theorem applies in particular to the special case where $P = A\Gamma$, $\rho = id_{A\Gamma}$ (see [L2] for the proof that $A\Gamma$ is a Hamiltonian H-space). In fact, we have a factorization method for solving the generalized Lax equations

$$\begin{aligned} &\frac{d}{dt}\,(q,0,X) \\ &= (\,\Pi_{\mathfrak{h}}\, df(X), 0, [X, R(q)df(X)]\,) \end{aligned} \tag{2.2.2}$$

on the invariant manifold $\gamma^{-1}(0)$. In what follows, if $(u,g,v) \in \Gamma$, the symbol $\boldsymbol{l}_{(u,g,v)}$ will stand for left translation in Γ by (u,g,v), the lift of $Im(\mathcal{R}^{+},\mathcal{R}^{-}) \subset A\Gamma \underset{TU}{\oplus} A\Gamma$ to the groupoid level will be denoted by the same symbol, and $\mathbf{Ad}$ is the adjoint representation of Γ on its adjoint bundle $Ker\, a$, defined by $\mathbf{Ad}_{\gamma}(q,0,X) = (q',0,Ad_k X)$, for $\gamma = (q',k,q) \in \Gamma$. Lastly, if Y is a section of $A\Gamma$, the exponential $exp\,(tY) : U \longrightarrow \Gamma$ is defined by the formula $exp(tY)(q) = f_t(\epsilon(q))$, where f_t is the local flow generated by the left-invariant vector field $\overleftarrow{Y}$: $\overleftarrow{Y}(u,g,v) = T_{\epsilon(v)}\boldsymbol{l}_{(u,g,v)}Y(v)$. In particular, if we take Y to be the constant section $(0,0,\xi)$ of $A\Gamma$, $\xi \in \mathfrak{g}$, an easy computation shows that $exp\{t(0,0,\xi)\}(q) = (q, e^{t\xi}, q)$.

Theorem 2.2.2. *Suppose that $f \in I(\mathfrak{g})$, $F = Pr_3^{*}f$ and $q_0 \in U$, where U is simply connected. Then for some $0 < T \leq \infty$, there exists a unique element $(\gamma_+(t),\gamma_-(t)) = ((q_0,k_+(t),q(t)),(q_0,k_-(t),q(t))) \in Im(\mathcal{R}^{+},\mathcal{R}^{-}) \subset \Gamma \underset{U\times U}{\times} \Gamma$ for $0 \leq t < T$ which is smooth in t, solves the factorization problem*

$$exp\{2\,t(0,0,df(X_0))\}(q_0) = \,\gamma_+(t)\,\gamma_-(t)^{-1} \tag{2.2.3}$$

and satisfies

$$\Big(T_{\gamma_+(t)}\boldsymbol{l}_{\gamma_+(t)^{-1}}\dot{\gamma}_+(t), T_{\gamma_-(t)}\boldsymbol{l}_{\gamma_-(t)^{-1}}\dot{\gamma}_-(t)\Big) \in (\mathcal{R}^{+},\mathcal{R}^{-})(\{q(t)\}\times\{0\}\times\mathfrak{g}) \tag{2.2.4a}$$

with

$$\gamma_{\pm}(0) = (q_0,1,q_0). \tag{2.2.4b}$$

Moreover, the solution of Eqn.(2.2.2) with initial data $(q,0,X)(0) = (q_0,0,X_0)$ is given by the formula

$$(q(t),0,X(t)) = \mathbf{Ad}_{\gamma_{\pm}(t)^{-1}}(q_0,0,X_0). \tag{2.2.5}$$

Corollary 2.2.3. *Let ψ_t be the induced flow on $\gamma^{-1}(0)$ as defined in (2.2.5) and let ϕ_t be the Hamiltonian flow of $\mathcal{F} = L^{*}f$ on P, where $L = Pr_3\circ\rho$ for a realization map $\rho : P \longrightarrow A\Gamma$ satisfying A1-A3. If we can solve for $\phi_t(x)$, $x \in J^{-1}(\mu)$ explicitly from the relation $\rho(\phi_t)(x) = \psi_t(\rho(x))$, then the formula $\phi_t^{red}\circ\pi_\mu = \pi_\mu\circ\phi_t\circ i_\mu$ gives an explicit expression for the flow of the reduced Hamiltonian $\mathcal{F}_\mu$.*

2.3 Classical dynamical r-matrices with spectral parameter and the associated spin Calogero-Moser systems.

From now onwards, we let $\mathfrak{g}$ be a complex simple Lie algebra of rank N with Killing form $(\cdot,\cdot)$ and let G be the connected and simply-connected Lie group which integrates $\mathfrak{g}$. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. For each $\alpha \in \Delta$, denote by H_α the element in $\mathfrak{h}$ which corresponds to α under the isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$ induced by the Killing form $(\cdot,\cdot)$. We fix a simple system of roots $\pi = \{\alpha_1, \cdots, \alpha_N\}$ and denote by $\Delta^\pm$ the corresponding positive/negative system. Also, for each $\alpha \in \Delta^+$, we pick root vectors $e_\alpha \in \mathfrak{g}_\alpha$, $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ which are dual with respect to $(\cdot,\cdot)$ so that $[e_\alpha, e_{-\alpha}] = H_\alpha$.

Let $r : \mathfrak{h}\times\mathbb{C} \longrightarrow \mathfrak{g}\otimes\mathfrak{g}$ be a classical dynamical r-matrix with spectral parameter in the sense of [EV], with coupling constant equal to 1. Then we can construct the associated classical dynamical r-matrix R [LX2] for the pair $(L\mathfrak{g}, \mathfrak{h})$, where $L\mathfrak{g}$ is the loop algebra of $\mathfrak{g}$. Indeed, if $L\mathfrak{g}^*$ denotes the restricted dual of $L\mathfrak{g}$ and we make the identification $L\mathfrak{g}^* \simeq L\mathfrak{g}$, then we have the following result.

Proposition 2.3.1. *For $X \in L\mathfrak{g}$, we have the formula*

$$(R(q)X)(z) = \frac{1}{2}X(z) + \sum_{k\geq 0} \frac{1}{k!}\left(\frac{\partial^k r}{\partial z^k}(q,-z),\ X_{-(k+1)}\otimes 1\right). \tag{2.3.1}$$

Moreover, R is a solution of the mDYBE with $K = \frac{1}{2}id_{L\mathfrak{g}}$.

Remark 2.3.2. As the reader will see, the formula in (2.3.1) will be used to compute the explicit expressions for R which play a critical role in characterizing the elements in the Lie subalgebroids $Im\mathcal{R}^\pm$ in our analysis in Section 3, 4 and 5 below. This formula has also been used in connection with symmetric coboundary dynamical Lie algebroids in [L3].

We now fix a simply-connected domain $U \subset \mathfrak{h}$ on which R is holomorphic. Also, introduce the trivial Lie groupoids $\Omega = U\times G\times U$, $\Gamma = U\times LG\times U$, where LG is the loop group of the simple Lie group G. Then we can use the map $R : U \longrightarrow L(L\mathfrak{g}, L\mathfrak{g})$ in Proposition 2.3.1 to construct the associated coboundary dynamical Lie algebroid $A^*\Gamma = T^*U \times L\mathfrak{g}^* \simeq TU \times L\mathfrak{g}$ so that its dual bundle has a Lie-Poisson structure. (See [L2] for explicit formulas.) On the other hand, we shall equip the dual bundle $A^*\Omega \simeq TU \times \mathfrak{g}$ of the trivial Lie algebroid $A\Omega$ with the corresponding Lie-Poisson structure. Explicitly, $\{\varphi,\psi\}_{A^*\Omega}(q,p,\xi) = (\delta_2\varphi, \delta_1\psi) - (\delta_1\varphi, \delta_2\psi) + (\xi, [\delta\varphi, \delta\psi])$ where $\delta_1\varphi$, $\delta_2\varphi$ and $\delta\varphi$ are the partial derivatives of φ with respect to the variables in U, $\mathfrak{h}$ and $\mathfrak{g}$ respectively.

Theorem 2.3.3. *The map $\rho = (m, \tau, L) : A^*\Omega \simeq TU \times \mathfrak{g} \longrightarrow TU \times L\mathfrak{g} \simeq A\Gamma$ given by*

$$\rho(q, p, \xi) = (q, -\Pi_{\mathfrak{h}}\xi, p + r_-^{\#}(q)\xi) \tag{2.3.2}$$

is an H-equivariant Poisson map, when the domain is equipped with the Lie-Poisson structure corresponding to the trivial Lie algebroid $A\Omega \simeq TU \times \mathfrak{g}$, and the target is equipped with the Lie-Poisson structure corresponding to $A^*\Gamma$. Here, the map $r_-^{\#}(q) : \mathfrak{g} \longrightarrow L\mathfrak{g}$ is defined by

$$((r_-^{\#}(q)\xi)(z), \eta) = (r(q, z), \eta \otimes \xi) \tag{2.3.3}$$

for ξ, $\eta \in \mathfrak{g}$.

Let Q be the quadratic function

$$Q(X) = \frac{1}{2} \oint_c (X(z), X(z)) \frac{dz}{2\pi i z} \tag{2.3.4}$$

where c is a small circle around the origin. Clearly, Q is an ad-invariant function on $L\mathfrak{g}$.

Definition 2.3.4. Let r be a classical dynamical r-matrix with spectral parameter with coupling constant equal to 1. Then the Hamiltonian system on $A^*\Omega \simeq TU \times \mathfrak{g}$ (equipped with the Lie-Poisson structure as in Theorem 2.3.3) generated by the Hamiltonian

$$\mathcal{H}(q, p, \xi) = (L^*Q)(q, p, \xi) = \frac{1}{2} \oint_c (L(q, p, \xi)(z), L(q, p, \xi)(z)) \frac{dz}{2\pi i z} \tag{2.3.5}$$

is called the spin Calogero-Moser system associated to r.

Proposition 2.3.5. *The Hamiltonians of the spin Calogero-Moser systems are invariant under the Hamiltonian H-action on $A^*\Omega \simeq TU \times \mathfrak{g}$ given by $h \cdot (q, p, \xi) = (q, p, Ad_h \xi)$ with equivariant momentum map $J(q, p, \xi) = -\Pi_{\mathfrak{h}}\, \xi$. Moreover, under the Hamiltonian flow, we have*

$$\begin{aligned} \dot{q} &= \Pi_{\mathfrak{h}}(M(q, p, \xi))_{-1}, \\ \dot{L}(q, p, \xi) &= [\, L(q, p, \xi), R(q)M(q, p, \xi)\,] \end{aligned} \tag{2.3.6}$$

on the invariant manifold $J^{-1}(0)$, where

$$M(q, p, \xi)(z) = L(q, p, \xi)(z)/z. \tag{2.3.7}$$

Clearly, the second part of the above proposition is a consequence of Theorem 2.2.1 (b) and Theorem 2.3.3.

In order to discuss the associated integrable models, we have to restrict to a smooth component of the reduced Poisson variety $J^{-1}(0)/H = U \times \mathfrak{h} \times (\mathfrak{h}^{\perp}/H)$. For this purpose, we restrict to the following open submanifold of $\mathfrak{g}$:

$$\mathcal{U} = \{\, \xi \in \mathfrak{g} \mid \xi_{\alpha_i} = (\xi, e_{-\alpha_i}) \neq 0, \quad i = 1, \ldots, N \,\}. \tag{2.3.8}$$

Then the H-action in Proposition 2.3.5 above induces a Hamiltonian H-action on $TU \times \mathcal{U}$ and we denote the corresponding momentum map also by J so that $J^{-1}(0) = TU \times (\mathfrak{h}^{\perp} \cap \mathcal{U})$. Now, recall from [LX2] that the formula

$$g(\xi) = \exp\left(\sum_{i=1}^{N} \sum_{j=1}^{N} (C_{ji} \log \xi_{\alpha_j}) h_{\alpha_i} \right) \tag{2.3.9}$$

defines an H-equivariant map $g : \mathcal{U} \longrightarrow H$ where $C = (C_{ij})$ is the inverse of the Cartan matrix and $h_{\alpha_i} = \frac{2}{(\alpha_i, \alpha_i)} H_{\alpha_i}$, $i = 1, \ldots, N$. Using g, we can identify the reduced space $J^{-1}(0)/H = TU \times (\mathfrak{h}^{\perp} \cap \mathcal{U}/H)$ with $TU \times \mathfrak{g}_{red}$, where $\mathfrak{g}_{red} = \epsilon + \sum_{\alpha \in \Delta - \pi} \mathbb{C} e_{\alpha}$, and $\epsilon = \sum_{j=1}^{N} e_{\alpha_j}$. Thus the projection map $\pi_0 : J^{-1}(0) \longrightarrow TU \times \mathfrak{g}_{red}$ is the map

$$(q, p, \xi) \mapsto (q, p, Ad_{g(\xi)^{-1}} \xi). \tag{2.3.10}$$

We shall write $s = \sum_{\alpha \in \Delta} s_{\alpha} e_{\alpha}$ for $s \in \mathfrak{g}_{red}$ (note that $s_{\alpha_j} = 1$ for $j = 1, \ldots, N$). By Poisson reduction [MR], the reduced manifold $TU \times \mathfrak{g}_{red}$ has a unique Poisson structure which is a product structure where the second factor $\mathfrak{g}_{red}$ is equipped with the reduction (at 0) of the Lie-Poisson structure on $\mathcal{U}$ by the H-action. If $\mathcal{H}$ is the Hamiltonian defined in (2.3.5), we shall denote its reduction to $TU \times \mathfrak{g}_{red}$ by $\mathcal{H}_0$.

3. The rational spin Calogero-Moser systems.

3.1. Lax operators, Hamiltonian equations and the Lie subalgebroids.

Let $\mathfrak{g}$ be a complex simple Lie algebra, as in Section 2.3. In addition to the basis $\{e_{\alpha}\}_{\alpha \in \Delta}$ of $\sum_{\alpha \in \Delta} g_{\alpha}$ in that section, let us now fix an orthonormal basis $(x_i)_{1 \leq i \leq N}$ of $\mathfrak{h}$. Thus we will write $p = \sum_i p_i x_i$, $\xi = \sum_i \xi_i x_i + \sum_{\alpha \in \Delta} \xi_{\alpha} e_{\alpha}$ for $p \in \mathfrak{h}$ and $\xi \in \mathfrak{g}$.

The rational spin Calogero-Moser systems are the Hamiltonian systems on $TU\times\mathfrak{g}$ (as defined in Definition 2.3.4) associated to the rational dynamical r-matrices with spectral parameter:

$$r(q,z)=\frac{\Omega}{z}+\sum_{\alpha\in\Delta'}\frac{1}{\alpha(q)}e_\alpha\otimes e_{-\alpha}, \tag{3.1.1}$$

where $\Delta'\subset\Delta$ is any set of roots which is closed with respect to addition and multiplication by -1, and $\Omega\in(S^2\mathfrak{g})^{\mathfrak{g}}$ is the Casimir element corresponding to the Killing form $(\cdot,\cdot)$. Accordingly, the Lax operators are given by

$$L(q,p,\xi)(z)=p+\sum_{\alpha\in\Delta'}\frac{\xi_\alpha}{\alpha(q)}e_\alpha+\frac{\xi}{z} \tag{3.1.2}$$

and so we have a family of Hamiltonians parametrized by Δ':

$$\mathcal{H}(q,p,\xi)=\frac{1}{2}\sum_i p_i^2-\frac{1}{2}\sum_{\alpha\in\Delta'}\frac{\xi_\alpha\xi_{-\alpha}}{\alpha(q)^2}. \tag{3.1.3}$$

Note that in particular, we have

$$L(q,p,\xi)(\infty)\in\mathfrak{g}_{\Delta'} \tag{3.1.4}$$

where

$$\mathfrak{g}_{\Delta'}=\mathfrak{h}+\sum_{\alpha\in\Delta'}\mathfrak{g}_\alpha \tag{3.1.5}$$

is a reductive Lie subalgebra of $\mathfrak{g}$. As the reader will see, this fact is important later on, when we solve the factorization problem.

Proposition 3.1.1. *The Hamiltonian equations of motion generated by $\mathcal{H}$ on $TU\times\mathfrak{g}$ are given by*

$$\begin{aligned}\dot q&=p,\\ \dot p&=-\sum_{\alpha\in\Delta'}\frac{\xi_\alpha\xi_{-\alpha}}{\alpha(q)^3}H_\alpha,\\ \dot\xi&=\left[\xi,-\sum_{\alpha\in\Delta'}\frac{\xi_\alpha}{\alpha(q)^2}e_\alpha\right].\end{aligned} \tag{3.1.6}$$

Proof. From the expression

$$\{\varphi,\psi\}(q,p,\xi)=(\delta_2\varphi,\delta_1\psi)-(\delta_1\varphi,\delta_2\psi)+(\xi,[\delta\varphi,\delta\psi])$$

for the Poisson bracket on $TU \times \mathfrak{g}$, the equations of motion are given by $\dot{q} = \delta_2\mathcal{H}$, $\dot{p} = -\delta_1\mathcal{H}$, and $\dot{\xi} = [\xi, \delta\mathcal{H}]$. Therefore, (3.1.6) follows by a direct computation. $\square$

We shall solve these equations on $J^{-1}(0)$ by our factorization method. To do so, it is essential to have the explicit expression of the classical dynamical r-matrix R associated to r. Before we make the computation, let us recall that the loop algebra $L\mathfrak{g}$ admits a direct sum decomposition

$$L\mathfrak{g} = L^{+}\mathfrak{g} \oplus L_0^{-}\mathfrak{g} \tag{3.1.7}$$

into Lie subalgebras, where $L^{+}\mathfrak{g}$ consists of convergent power series $\sum_0^\infty X_n z^n$, while $L_0^{-}\mathfrak{g}$ consists of Laurent tails, of the form $\sum_{-\infty}^{-1} X_n z^n$. We shall denote by $\Pi_\pm$ the projection operators relative to this splitting.

Proposition 3.1.2. *The classical dynamical r-matrix R associated with the meromorphic map r in (3.1.1) is given by*

$$(R(q)X)(z) = \frac{1}{2}\big(\Pi_+ X - \Pi_- X\big)(z) - \sum_{\alpha\in\Delta'} \frac{(X_{-1})_\alpha}{\alpha(q)} e_\alpha. \tag{3.1.8}$$

In particular,

$$(R(q)M(q,p,\xi))(z) = -\frac{1}{2}M(q,p,\xi)(z) - \sum_{\alpha\in\Delta'} \frac{\xi_\alpha}{\alpha(q)^2} e_\alpha \tag{3.1.9}$$

for $M(q,p,\xi)(z) = L(q,p,\xi)(z)/z$. (See (2.3.6),(2.3.7).)

Proof. By direct differentiation, we find that $\frac{\partial^k r}{\partial z^k}(q,-z) = -k!\frac{\Omega}{z^{k+1}}$, $k \geq 1$. Substituting into (2.3.1), the formula follows. $\square$

Remark 3.1.3. (a) The formula in (3.1.8) shows that the classical dynamical r-matirx R is a perturbation of the standard r-matrix associated with the splitting in (3.1.7).

(b) If we restrict ourselves to $J^{-1}(0)$, then by equating the coefficients of z^0 and z^{-1} on both sides of the Lax equation $\dot{L}(q,p,\xi) = [\,L(q,p,\xi), R(q)M(q,p,\xi)\,]$, we can recover the equations for p and ξ respectively in (3.1.6). However, the Lax equation only gives $\alpha(\dot{q} - p)$ for all $\alpha \in \Delta'$. Therefore, unless $\Delta' = \Delta$, otherwise, we cannot recover the equation for q from that of $L(q,p,\xi)$. This remark shows that the full set of equations in (2.3.6) is important.

We now give the equations of motion for the reduction of $\mathcal{H}$ on $TU \times \mathfrak{g}_{red}$, with Hamiltonian given by

$$\mathcal{H}_0(q,p,s) = \frac{1}{2}\sum_i p_i^2 - \frac{1}{2}\sum_{\alpha\in\Delta'} \frac{s_\alpha s_{-\alpha}}{\alpha(q)^2}. \tag{3.1.10}$$

Proposition 3.1.4. *The Hamiltonian equations of motion generated by $\mathcal{H}_0$ on the reduced Poisson manifold $TU \times \mathfrak{g}_{red}$ are given by*

$$\dot{q} = p,$$
$$\dot{p} = -\sum_{\alpha\in\Delta'} \frac{s_\alpha s_{-\alpha}}{\alpha(q)^3} H_\alpha,$$

$$\dot{s} = [\, s, \mathcal{M}\,]$$

where

$$\mathcal{M} = -\sum_{\alpha\in\Delta'} \frac{s_\alpha}{\alpha(q)^2} e_\alpha + \sum_{i,j} C_{ji} \sum_{\substack{\alpha\in\Delta' \\ \alpha_j-\alpha\in\Delta}} N_{\alpha,\alpha_j-\alpha} \frac{s_\alpha s_{\alpha_j-\alpha}}{\alpha(q)^2} h_{\alpha_i}.$$

(Here we use the notation $[e_\alpha, e_\beta] = N_{\alpha,\beta}\, e_{\alpha+\beta}$ if $\alpha+\beta \in \Delta$.)

Proof. The equations for q and p are obvious from Propostion 3.1.1 and the fact that $s_\alpha = \xi_\alpha e^{-\alpha(log g(\xi))}$. To derive the equation for s, we differentiate $s = Ad_{g(\xi)^{-1}}\xi$ with respect to t, assuming that ξ satisfies the equation in Proposition 3.1.1 with $\Pi_{\mathfrak{h}}\xi = 0$. This gives

$$\dot{s} = \left[s, -\sum_{\alpha\in\Delta'} \frac{s_\alpha}{\alpha(q)^2} e_\alpha - T_{g(\xi)^{-1}} r_{g(\xi)} \frac{d}{dt} g(\xi)^{-1} \right]. \qquad (*)$$

Now, from the expression for $g(\xi)$ in (2.3.9), we find

$$-T_{g(\xi)^{-1}} r_{g(\xi)} \frac{d}{dt} g(\xi)^{-1} = \sum_{i,j} C_{ji} \dot{\xi}_{\alpha_j} {\xi_{\alpha_j}}^{-1} h_{\alpha_i}.$$

But from Proposition 3.1.1, we have

$$\begin{aligned} \dot{\xi}_{\alpha_j} &= \left(\left[\xi, -\sum_{\alpha\in\Delta'} \frac{\xi_\alpha}{\alpha(q)^2} e_\alpha \right], e_{-\alpha_j} \right) \\ &= \xi_{\alpha_j} \sum_{\substack{\alpha\in\Delta' \\ \alpha_j-\alpha\in\Delta}} N_{\alpha,\alpha_j-\alpha} \frac{s_\alpha s_{\alpha_j-\alpha}}{\alpha(q)^2}. \end{aligned}$$

Therefore, on substituting the above expressions into (*), we obtain the desired equation for s. □

In order to solve the equations in (2.3.6) by the factorization method, it is necessary to have precise description of the Lie algebroids and Lie groupoids which are involved. We now begin to describe these geometric objects. Let $L^-\mathfrak{g}$ be the

Lie subalgebra of $L\mathfrak{g}$ consisting of series of the form $\sum_{-\infty}^{0} X_n z^n$. From the explicit expression for R in (3.1.8), we have

$$(R^{\pm}(q)X)(z) = \pm(\Pi_{\pm}X)(z) - \sum_{\alpha\in\Delta'} \frac{(X_{-1})_\alpha}{\alpha(q)} e_\alpha. \tag{3.1.11}$$

Therefore, $R^+(q)X \in L^+\mathfrak{g}$, while $R^-(q)X \in L^-_{\Delta'}\mathfrak{g}$, where

$$L^-_{\Delta'}\mathfrak{g} = \{X \in L^-\mathfrak{g} \mid X(\infty) \in \mathfrak{g}_{\Delta'}\}. \tag{3.1.12}$$

The proof of the next proposition is obvious and will be left to the reader.

Proposition 3.1.5. *(a)* $Im\mathcal{R}^+ = \bigcup_{q\in U}\{0_q\} \times L^+\mathfrak{g} \times \mathfrak{h}$.
(b) $\mathcal{I}^+ = \bigcup_{q\in U}\{0_q\} \times L^+\mathfrak{g} \times \{0\}$ *= adjoint bundle of* $Im\mathcal{R}^+$.

Remark 3.1.6. Indeed, we also have $\{\mathcal{R}^+(0_q, X, 0) \mid q \in U, X \in L\mathfrak{g}\} = \bigcup_{q\in U}\{0_q\}\times L^+\mathfrak{g} \times \mathfrak{h}$.

Before we turn to the characterization of $Im\mathcal{R}^-$, let us recall the notion of a matched pair of Lie algebroids introduced in [Mok] as an infinitesimal version of the notion of a matched pair of Lie groupoids [M2].(These are generalizations of the corresponding notions for Lie algebras and Lie groups, see [KSM], [LW],[Maj].)

Definition 3.1.7. Two Lie algebroids A_1, A_2 over the base B is said to form a matched pair of Lie algebroids iff the Whitney sum $W = A_1 \oplus A_2$ admits a Lie algebroid structure over the same base with A_1 and A_2 as Lie subalgebroids. In this case, the Lie algebroid W is called the matched product of A_1 and A_2 and is denoted by $A_1 \bowtie A_2$.

Proposition 3.1.8. *(a) The ideal* $\mathcal{I}^-$ *is given by*

$$\mathcal{I}^- = \left\{(0_q, X, 0) \;\middle|\; q \in U, X \in L^-_{\Delta'}\mathfrak{g},\ X_{-1} \in \mathfrak{h}^\perp \text{ and } \sum_{\alpha\in\Delta'} (X_{-1})_\alpha e_\alpha = ad_q \Pi_{\mathfrak{h}^\perp} X_0\right\}.$$

(b) $Im\mathcal{R}^- = \mathcal{I}^- \bowtie \mathcal{Q}$*, where*

$$\mathcal{Q} = \left\{(0_q, -\widetilde{Z}, Z) \mid q \in U, Z \in \mathfrak{h},\ \text{and}\ \widetilde{Z}(z) = Zz^{-1}\right\}$$

is a Lie subalgebroid of $Im\mathcal{R}^-$*. Hence* $\mathcal{I}^-$ *coincides with the adjoint bundle of the transitive Lie algebroid* $Im\mathcal{R}^-$*. Moreover,* $\mathcal{R}^-(\{0_q\}\times L\mathfrak{g}\times\{0\})$ *can be characterized as the set*

$$\left\{(0_q, X, Z) \mid Z \in \mathfrak{h}, X \in L^-_{\Delta'}\mathfrak{g}, \Pi_{\mathfrak{h}} X_0 = 0 \text{ and } \Pi_{\mathfrak{g}_{\Delta'}} X_{-1} = -Z + ad_q X_0\right\}$$

where $\Pi_{\mathfrak{g}_{\Delta'}}$ *is the projection map relative to the decomposition* $\mathfrak{g} = \mathfrak{g}_{\Delta'} \oplus (\mathfrak{g}_{\Delta'})^{\perp}$.

Proof. (a) From the definition of $\mathcal{I}^-$ in (2.1.10) and the expression for $R^+(q)$ in (3.1.11), we have

$$
\begin{aligned}
&(0_q, X, 0) \in \mathcal{I}^- \\
\Longleftrightarrow\ & \mathcal{R}^+(0_q, X, Z) = 0 \text{ for some } Z \in \mathfrak{h} \\
\Longleftrightarrow\ & \Pi_{\mathfrak{h}} X_{-1} = 0,\ -\iota Z + (\Pi_+ X)(z) - \sum_{\alpha \in \Delta'} \frac{(X_{-1})_\alpha}{\alpha(q)} e_\alpha = 0 \text{ for some } Z \in \mathfrak{h} \\
\Longleftrightarrow\ & X \in L^-_{\Delta'}\mathfrak{g},\ X_{-1} \in \mathfrak{h}^{\perp} \text{ and } \sum_{\alpha \in \Delta'} (X_{-1})_\alpha e_\alpha = ad_q \Pi_{\mathfrak{h}^\perp} X_0.
\end{aligned}
$$

Hence the assertion.

(b) For an element $Z \in \mathfrak{h}$, let $\widetilde{Z}$ be the loop given by $\widetilde{Z}(z) = Zz^{-1}$. Consider an arbitrary element $\mathcal{R}^-(0_q, X, Z)$ in $Im\mathcal{R}^-$. Clearly, it admits the decomposition

$$
\begin{aligned}
&\mathcal{R}^-(0_q, X, Z) \\
=\ & (0_q, -\iota Z - \Pi_- X + \widetilde{\Pi_{\mathfrak{h}} X_{-1}} - \sum_{\alpha \in \Delta'} \frac{(X_{-1})_\alpha}{\alpha(q)} e_\alpha, 0) \\
&+ (0_q, -\widetilde{\Pi_{\mathfrak{h}} X_{-1}}, \Pi_{\mathfrak{h}} X_{-1})
\end{aligned}
$$

where the first term is in $\mathcal{I}^-$ and the second term is in $\mathcal{Q}$. This shows that $Im\mathcal{R}^- \subset \mathcal{I}^- \oplus \mathcal{Q}$. Conversely, take an arbitrary element $(0_q, X, 0) + (0_q, -\widetilde{Z}, Z) \in \mathcal{I}^- \oplus \mathcal{Q}$ and let $Y \in L\mathfrak{g}$ be defined by $Y = -\Pi_- X + \widetilde{Z}$. Then from the characterization of $\mathcal{I}^-$ in part (a), we have

$$
\begin{aligned}
&\mathcal{R}^-(0_q, Y, -\Pi_{\mathfrak{h}} X_0) \\
=\ & (0_q, \Pi_{\mathfrak{h}} X_0 - \Pi_- Y - \sum_{\alpha \in \Delta'} \frac{(Y_{-1})_\alpha}{\alpha(q)} e_\alpha, \Pi_{\mathfrak{h}} Y_{-1}) \\
=\ & (0_q, \Pi_{\mathfrak{h}} X_0 + \Pi_- X - \widetilde{Z} + \sum_{\alpha \in \Delta'} (X_0)_\alpha e_\alpha, Z) \\
=\ & (0_q, X, 0) + (0_q, -\widetilde{Z}, Z)
\end{aligned}
$$

and this shows $\mathcal{I}^- \oplus \mathcal{Q} \subset Im\mathcal{R}^-$. Combining the two inclusions, we conclude that $Im\mathcal{R}^- = \mathcal{I}^- \bowtie \mathcal{Q}$. We shall leave the details of the other assertions to the reader. □

As a consequence of Proposition 3.1.5, we have

$$
\begin{aligned}
Im\mathcal{R}^+/\mathcal{I}^+ &= \bigcup_{q \in U} \{0_q\} \times (L^+\mathfrak{g}/L^+\mathfrak{g}) \times \mathfrak{h} \\
&\simeq \bigcup_{q \in U} \{0_q\} \times \{0\} \times \mathfrak{h}
\end{aligned}
\tag{3.1.13}
$$

where the identification map is given by

$$(0_q, X + L^+\mathfrak{g}, Z) \mapsto (0_q, 0, Z). \tag{3.1.14}$$

Similarly, it follows from Proposition 3.1.8 that

$$Im\mathcal{R}^-/\mathcal{I}^- \simeq \mathcal{Q} \tag{3.1.15}$$

and the identification map is

$$(0_q, X, Z) + \mathcal{I}_q^- \mapsto (0_q, -\widetilde{Z}, Z). \tag{3.1.16}$$

The following proposition is obvious.

Proposition 3.1.9. *The isomorphism* $\theta : Im\mathcal{R}^+/\mathcal{I}^+ \longrightarrow Im\mathcal{R}^-/\mathcal{I}^-$ *defined in Proposition 2.1.2 (b) is given by*

$$\theta(0_q, 0, Z) = (0_q, -\widetilde{Z}, Z).$$

Moreover, $Im(\mathcal{R}^+, \mathcal{R}^-) = Im\mathcal{R}^+ \underset{TU}{\oplus} Im\mathcal{R}^-$.

3.2. Solution of the integrable rational spin Calogero-Moser systems.

We begin by solving the equation

$$\begin{aligned}&\frac{d}{dt}(q, 0, L(q,p,\xi))\\ &= (p, 0, [\, L(q,p,\xi), R(q)M(q,p,\xi)\,])\end{aligned} \tag{3.2.1}$$

where explicitly,

$$M(q,p,\xi)(z) = \frac{1}{z}\left(p + \sum_{\alpha\in\Delta'} \frac{\xi_\alpha}{\alpha(q)} e_\alpha\right) + \frac{\xi}{z^2}. \tag{3.2.2}$$

To do so, we have to solve the factorization problem

$$exp\{\, t(0, 0, M(q^0, p^0, \xi^0))\}(q^0) = \gamma_+(t)\, \gamma_-(t)^{-1} \tag{3.2.3}$$

for $(\gamma_+(t), \gamma_-(t)) = ((q^0, k_+(t), q(t)), (q^0, k_-(t), q(t))) \in Im(\mathcal{R}^+, \mathcal{R}^-)$ satisfying the condition in (2.2.4), where $(q^0, p^0, \xi^0) \in J^{-1}(0) = TU \times (\mathcal{U} \cap \mathfrak{h}^\perp)$ is the initial value of (q, p, ξ). In what follows, we shall denote by LG, L^+G, L_1^-G and $L_{\Delta'}^-G$ the loop groups corresponding to the Lie algebras $L\mathfrak{g}$, $L^+\mathfrak{g}$, $L_0^-\mathfrak{g}$ and $L_{\Delta'}^-\mathfrak{g}$ respectively. We shall also denote $k_\pm(t)(z)$ by $k_\pm(z,t)$. Then $k_+(\cdot, t) \in L^+G$, while $k_-(\cdot, t) \in L_{\Delta'}^-G$

and satisfies additional constraints. From the factorization problem on the Lie groupoid above, it follows that

$$e^{tM(q^0,p^0,\xi^0)(z)} = k_+(z,t)k_-(z,t)^{-1} \tag{3.2.4}$$

where $k_\pm(\cdot,t)$ are to be determined. To do so, we recall from the Birkhoff factorization theorem [PS] that (at least for small values of t)

$$e^{tM(q^0,p^0,\xi^0)(z)} = g_+(z,t)g_-(z,t)^{-1} \tag{3.2.5}$$

for unique $g_+(\cdot,t) \in L^+G$ and $g_-(\cdot,t) \in L_1^-G$. But from (3.2.2) above, it is clear that $e^{tM(q^0,p^0,\xi^0)} \in L_1^-G$, so we must have (i) $g_+ \equiv 1$, (ii) $g_-(z,t) = e^{-tM(q^0,p^0,\xi^0)}(z)$ for all t and consequently,

$$k_+(z,t) \equiv k_-(\infty,t). \tag{3.2.6}$$

Thus we have the relation

$$e^{tM(q^0,p^0,\xi^0)(z)} = k_-(\infty,t)k_-(z,t)^{-1} \tag{3.2.7}$$

where $\gamma_-(t) = (q^0, k_-(t), q(t))$ is subject to the condition $T_{\gamma_-(t)}\boldsymbol{l}_{\gamma_-(t)^{-1}}\dot{\gamma}_-(t) \in \mathcal{R}^-(\{q(t)\}\times\{0\}\times L\mathfrak{g})$. But from the characterization of $\mathcal{R}^-(\{q(t)\}\times\{0\}\times L\mathfrak{g})$ in Proposition 3.1.8 (b), we have

$$\begin{aligned} &\Pi_{\mathfrak{g}_{\Delta'}} Res_{z=0}\, T_{k_-(z,t)}l_{k_-(z,t)^{-1}}\dot{k}_-(z,t) \\ =& -\dot{q}(t) + ad_{q(t)}T_{k_-(\infty,t)}l_{k_-(\infty,t)^{-1}}\dot{k}_-(\infty,t). \end{aligned} \tag{3.2.8}$$

On the other hand, by differentiating (3.2.7) with respect to t, we find

$$\begin{aligned} &T_{k_-(z,t)}l_{k_-(z,t)^{-1}}\dot{k}_-(z,t) - T_{k_-(\infty,t)}l_{k_-(\infty,t)^{-1}}\dot{k}_-(\infty,t) \\ =& -Ad_{k_-(\infty,t)^{-1}}M(q^0,p^0,\xi^0)(z) \end{aligned}$$

from which it follows that

$$Res_{z=0}\, T_{k_-(z,t)}l_{k_-(z,t)^{-1}}\dot{k}_-(z,t) = -Ad_{k_-(\infty,t)^{-1}}\left(p^0 + \sum_{\alpha\in\Delta'}\frac{\xi^0_\alpha}{\alpha(q^0)}e_\alpha\right). \tag{3.2.9}$$

Therefore, upon substituting (3.2.9) into (3.2.8), we obtain

$$\begin{aligned} &Ad_{k_-(\infty,t)}\,\dot{q}(t) + \left[T_{k_-(\infty,t)}r_{k_-(\infty,t)^{-1}}\dot{k}_-(\infty,t), Ad_{k_-(\infty,t)}\,q(t)\right] \\ =& L(q^0,p^0,\xi^0)(\infty), \end{aligned}$$

that is,

$$\frac{d}{dt} Ad_{k_-(\infty,t)}\, q(t) = L(q^0, p^0, \xi^0)(\infty). \tag{3.2.10}$$

Hence the factorization problem boils down to

$$q^0 + tL(q^0, p^0, \xi^0)(\infty) = Ad_{k_-(\infty,t)}\, q(t) \tag{3.2.11}$$

where $q(t)$ and $k_-(\infty, t)$ are to be determined. But from (3.1.4) and the fact that $\mathfrak{g}_{\Delta'}$ is reductive, we can find (at least for small values of t) unique $d(t) \in H$ and $g(t) \in G_{\Delta'}$ (unique up to $g(t) \to g(t)\delta(t)$, where $\delta(t) \in H$) such that

$$q^0 + tL(q^0, p^0, \xi^0) = Ad_{g(t)}\, d(t) \tag{3.2.12}$$

with $g(0) = 1$, $d(0) = q^0$. Hence

$$q(t) = d(t). \tag{3.2.13}$$

On the other hand, let us fix one such $g(t)$. We shall seek $k_-(\infty, t)$ in the form

$$k_-(\infty, t) = g(t)h(t), \quad h(t) \in H. \tag{3.2.14}$$

To determine $h(t)$, note that the characterization of $\mathcal{R}^-(\{q(t)\} \times \{0\} \times L\mathfrak{g})$ in Proposition 3.1.8 (b) also gives

$$\Pi_{\mathfrak{h}}\, T_{k_-(\infty,t)} l_{k_-(\infty,t)^{-1}} \dot{k}_-(\infty, t) = 0. \tag{3.2.15}$$

Using this condition, we find that $h(t)$ satisfies the equation

$$\dot{h}(t) = T_e l_{h(t)} \left(-\Pi_{\mathfrak{h}} (T_{g(t)} l_{g(t)^{-1}} \dot{g}(t)) \right) \tag{3.2.16}$$

with $h(0) = 1$. Solving the equation explicitly, we obtain

$$h(t) = exp \left\{ -\int_0^t \Pi_{\mathfrak{h}} (T_{g(\tau)} l_{g(\tau)^{-1}} \dot{g}(\tau))\, d\tau \right\}. \tag{3.2.17}$$

Hence $k_+(z, t) \equiv k_-(\infty, t)$ and $k_-(z, t) \equiv e^{-tM(q^0,p^0,\xi^0)(z)} k_-(\infty, t)$ satisfy (3.2.4).

Theorem 3.2.1. *Let $(q^0, p^0, \xi^0) \in J^{-1}(0) = TU \times (\mathcal{U} \cap \mathfrak{h}^\perp)$. Then the Hamiltonian flow on $J^{-1}(0)$ generated by*

$$\mathcal{H}(q, p, \xi) = \frac{1}{2} \sum_i p_i^2 - \frac{1}{2} \sum_{\alpha \in \Delta'} \frac{\xi_\alpha \xi_{-\alpha}}{\alpha(q)^2}$$

with initial condition $(q(0), p(0), \xi(0)) = (q^0, p^0, \xi^0)$ *is given by*

$$\begin{aligned} q(t) &= d(t), \\ \xi(t) &= Ad_{k_-(\infty,t)^{-1}}\xi^0, \\ p(t) &= \dot{d}(t) = Ad_{k_-(\infty,t)^{-1}} L(q^0, p^0, \xi^0)(\infty) - \sum_{\alpha\in\Delta'} \frac{\xi(t)_\alpha}{\alpha(q(t))} e_\alpha \end{aligned} \tag{3.2.18}$$

where $d(t)$ *and* $k_-(\infty, t)$ *are constructed from the above procedure.*

Proof. The formulas for $p(t)$, $\xi(t)$ are obtained by equating the coefficients of z^0 and z^{-1} on both sides of the expression $L(q(t), p(t), \xi(t))(z) = Ad_{k_-(\infty,t)^{-1}} L(q^0, p^0, \xi^0)(z)$. □

We now turn to the solution of the associated integrable model on $TU \times \mathfrak{g}_{red}$ with Hamiltonian $\mathcal{H}_0(q, p, s) = \frac{1}{2}\sum_i p^2 - \frac{1}{2}\sum_{\alpha\in\Delta'} \frac{s_\alpha s_{-\alpha}}{\alpha(q)^2}$ and with equations of motion given in Proposition 3.1.4.

Corollary 3.2.2. *Let* $(q^0, p^0, s^0) \in TU \times \mathfrak{g}_{red}$ *and suppose* $s^0 = Ad_{g(\xi^0)^{-1}}\xi^0$ *where* $\xi^0 \in \mathcal{U} \cap \mathfrak{h}^\perp$. *Then the Hamiltonian flow generated by* $\mathcal{H}_0$ *with initial condition* $(q(0), p(0), s(0)) = (q^0, p^0, s^0)$ *is given by*

$$\begin{aligned} q(t) &= d(t), \\ s(t) &= Ad_{\left(\widetilde{k}_-(\infty,t)\, g\left(Ad_{\widetilde{k}_-(\infty,t)^{-1}} s^o\right)\right)^{-1}} s^0, \\ p(t) &= Ad_{\left(\widetilde{k}_-(\infty,t)\, g\left(Ad_{\widetilde{k}_-(\infty,t)^{-1}} s^o\right)\right)^{-1}} L(q^0, p^0, s^0)(\infty) - \sum_{\alpha\in\Delta'} \frac{s_\alpha(t)}{\alpha(q(t))} e_\alpha. \end{aligned} \tag{3.2.19}$$

where $\widetilde{k}_-(\infty, t) = g(\xi^0)^{-1} k_-(\infty, t) g(\xi^0)$ *depends only on* s^0 *and* $k_-(\infty, t)$, $d(t)$ *are as in Theorem 3.2.1.*

proof. We shall obtain the Hamiltonian flow generated by $\mathcal{H}_0$ by Poisson reduction. Using the relation $\phi_t^{red} \circ \pi_0 = \pi_0 \circ \phi_t \circ i_0$ from Corollary 2.2.3, we have

$$\phi_t^{red}(q^0, p^0, s^0) = (q(t), p(t), Ad_{g(\xi(t))^{-1}}\xi(t))$$

where $q(t)$ and $p(t)$ are given by the expressions in Theorem 3.2.1 above. Thus

$$s(t) = Ad_{g(\xi(t))^{-1}}\xi(t) = Ad_{\left(\widetilde{k}_-(\infty,t)\, g\left(Ad_{\widetilde{k}_-(\infty,t)^{-1}} s^o\right)\right)^{-1}} s^0$$

where we have used the H-equivariance of the map g to show that

$$g(\xi^0)^{-1} k_-(\infty, t) g(\xi(t)) = \widetilde{k}_-(\infty, t)\, g\left(Ad_{\widetilde{k}_-(\infty,t)^{-1}} s^o\right).$$

To express $p(t)$ in the desired form, simply apply $Ad_{g(\xi(t))^{-1}}$ to both sides of the expression for $p(t)$ in the above theorem, this gives

$$p(t) = Ad_{g(\xi(t))^{-1}k_-(\infty,t)^{-1}g(\xi^0)}L(q^0,p^0,s^0)(\infty) - \sum_{\alpha\in\Delta'}\frac{s_\alpha(t)}{\alpha(q(t))}e_\alpha$$

where we have used the relation $s_\alpha(t) = e^{-\alpha(g(\xi(t)))}\xi_\alpha(t)$. Hence the desired expression for $p(t)$ follows. The assertion that $\widetilde{k}_-(\infty,t)$ depends only on s^0 is clear. □

Remark 3.2.3. (a) The expression in (3.2.12) shows that the solution blows up precisely when the factorization fails. However, initial conditions do exist for which the solution exists for all time.

(b) The first example of a rational spin Calogero-Moser system is due to Gibbons and Hermsen [GH]. Analogous to what was done there, we can show that

$$\dot{q} = L(q,p,\xi)(\infty) + \left[q\,, -\sum_{\alpha\in\Delta'}\frac{\xi_\alpha}{\alpha(q)^2}e_\alpha\right]$$

on $J^{-1}(0)$ from which we can also deduce the relation (3.2.12). Thus on the surface, it appears that there is no need to use $L(q,p,\xi)(z)$. We remark, however, that our factorization problem (which involves $L(q^0,p^0,\xi^0)(z)$) does carry more information and that we do need $L(q,p,\xi)(z)$ in order to establish the Liouville integrability of the reduced system on $TU\times\mathfrak{g}_{red}$. In other words, our realization picture embraces both exact solvability and complete integrability. A unifying and representation independent method to establish the Liouville integrability of the integrable spin CM systems in [LX2] for all simple Lie algebras will be given in a forthcoming paper.

(c) We now explain the Poisson meaning of the limiting Lax operator $L(q,p,\xi)(\infty)$. To do so, we recall that $r(q) = \sum_{\Delta'}\frac{1}{\alpha(q)}e_\alpha\otimes e_{-\alpha}$ is a classical dynamical r-matrix with zero coupling constant in the sense of [EV]. Therefore, if we define $R : U \longrightarrow L(\mathfrak{g},\mathfrak{g})$ by

$$R(q)\xi = r^\sharp(q)\xi = -\sum_{\alpha'}\frac{\xi_\alpha}{\alpha(q)}e_\alpha,$$

then R is a solution of the CDYBE (i.e., (2.1.2) with $\chi\equiv 0$). Let $A^*\Omega\simeq TU\times\mathfrak{g}$ be the coboundary dynamical Lie algebroid associated with R and let $A\Omega\simeq TU\times\mathfrak{g}$ be the trivial Lie algebroid. Then according to [L2],

$$\mathcal{R} : A^*\Omega \longrightarrow A\Omega, (q,p,\xi)\mapsto (q,\Pi_{\mathfrak{h}}\xi,-p+R(q)\xi)$$

is a morphism of Lie algebroids. Consequently, the dual map $\mathcal{R}^*$ is an H-equivariant Poisson map, when the domain and target are equipped with the corresponding Lie-Poisson structure. Explicitly,

$$\begin{aligned}\mathcal{R}^*(q,p,\xi) =&(q, -\Pi_{\mathfrak{h}}\xi, p - R(q)\xi)\\ =&(q, -\Pi_{\mathfrak{h}}\xi, L(q,p,\xi)(\infty)).\end{aligned}$$

Moreover, if we define

$$L^{\infty}(q,p,\xi) = L(q,p,\xi)(\infty),$$

then the Hamiltonian $\mathcal{H}$ of the rational spin CM system in (3.1.3) is also given by

$$\mathcal{H}(q,p,\xi) = ((L^{\infty})^* E)(q,p,\xi)$$

where E is the quadratic function on $\mathfrak{g}$ defined by

$$E(\xi) = \frac{1}{2}(\xi,\xi).$$

This shows that the Hamiltonian system defined by $\mathcal{H}$ admits a second realization in $A\Omega$ and this clarifies the Poisson-geometric meaning of $L(q,p,\xi)(\infty)$. We note, however, that the r-matrix $\mathcal{R}$ introduced earlier in this remark is degenerate in the sense that it is not associated with a factorization problem.

4. The trigonometric spin Calogero-Moser systems.

4.1. Lax operators, Hamiltonian equations and the Lie subalgebroids.

In this section, we take the trigonometric spin Calogero-Moser systems to be the Hamiltonian systems in Definition 2.3.4 associated to the following trigonometric dynamical r-matrices with spectral parameter:

$$r(q,z) = c(z)\sum_i x_i \otimes x_i - \sum_{\alpha\in\Delta} \phi_\alpha(q,z) e_\alpha \otimes e_{-\alpha} \tag{4.1.1}$$

where

$$c(z) = \cot z \tag{4.1.2}$$

and

$$\phi_\alpha(q,z) = \begin{cases} -\frac{\sin(\alpha(q)+z)}{\sin\alpha(q)\sin z}, & \alpha \in <\pi'> \\ -\frac{e^{-iz}}{\sin z}, & \alpha \in \overline{\pi}'^{+} \\ -\frac{e^{iz}}{\sin z}, & \alpha \in \overline{\pi}'^{-}. \end{cases} \tag{4.1.3}$$

In (4.1.3) above, π' is an arbitrary subset of a fixed simple system $\pi \subset \Delta$, $< \pi' >$ is the root span of π' and $\overline{\pi}'^{\pm} = \Delta^{\pm} \backslash < \pi' >^{\pm}$. Accordingly, the Lax operators are given by

$$
\begin{aligned}
L(q,p,\xi)(z) =& p + c(z)\sum_i \xi_i x_i - \sum_{\alpha\in\Delta} \phi_\alpha(q,z)\xi_\alpha e_\alpha \\
=& p + \sum_{\alpha\in<\pi'>} c(\alpha(q))\xi_\alpha - i\sum_{\alpha\in\overline{\pi}'^{+}} \xi_\alpha e_\alpha \\
&+ i\sum_{\alpha\in\overline{\pi}'^{-}} \xi_\alpha e_\alpha + c(z)\xi.
\end{aligned} \tag{4.1.4}
$$

Hence we have a family of dynamical systems parametrized by subsets π' of π with Hamiltonians of the form:

$$
\begin{aligned}
\mathcal{H}(q,p,\xi) =& \frac{1}{2}\sum_i p_i^2 - \frac{1}{2}\sum_{\alpha\in<\pi'>}\left(\frac{1}{\sin^2\alpha(q)} - \frac{1}{3}\right)\xi_\alpha\xi_{-\alpha} - \frac{5}{6}\sum_{\alpha\in\Delta\backslash<\pi'>}\xi_\alpha\xi_{-\alpha} \\
&- \frac{1}{3}\sum_i \xi_i^2.
\end{aligned} \tag{4.1.5}
$$

Remark 4.1.1. The trigonometric dynamical r-matrices in (4.1.1) are gauge equivalent to those used in [LX2]. Although the corresponding Hamiltonians in (4.1.5) above contain the additional term $-\frac{1}{3}\sum_i \xi_i^2$, however, the Hamiltonian flows on $J^{-1}(0)$ and the reduced systems are the same as those in [LX2]. The reason why we use the dynamical r-matrices in (4.1.1) is due to the fact that the corresponding Lie subalgebroids $Im\mathcal{R}^{\pm}$ are simpler to analyze.

The next two propositions follow from direct calculation, as in Propositions 3.1.1 and 3.1.4. We shall leave the proof to the reader.

Proposition 4.1.2. *The Hamiltonian equations of motion generated by $\mathcal{H}$ on $TU \times \mathfrak{g}$ are given by*

$$
\begin{aligned}
\dot{q} &= p, \\
\dot{p} &= -\sum_{\alpha\in<\pi'>}\frac{\cot\alpha(q)}{\sin^2\alpha(q)}\xi_\alpha\xi_{-\alpha}H_\alpha, \\
\dot{\xi} &= \left[\xi, -\frac{2}{3}\Pi_{\mathfrak{h}}\xi - \sum_{\alpha\in<\pi'>}\left(\frac{1}{\sin^2\alpha(q)} - \frac{1}{3}\right)\xi_\alpha e_\alpha - \frac{5}{3}\sum_{\alpha\in\Delta\backslash<\pi'>}\xi_\alpha e_\alpha\right].
\end{aligned} \tag{4.1.6}
$$

Proposition 4.1.3. *The Hamiltonian equations of motion generated by*

$$\mathcal{H}_0(q,p,s) = \frac{1}{2}\sum_i p_i^2 - \frac{1}{2}\sum_{\alpha\in<\pi'>}\left(\frac{1}{\sin^2\alpha(q)} - \frac{1}{3}\right)s_\alpha s_{-\alpha} - \frac{5}{6}\sum_{\alpha\in\Delta\setminus<\pi'>} s_\alpha s_{-\alpha}$$

on the reduced Poisson manifold $TU \times \mathfrak{g}_{red}$ are given by

$$\begin{aligned}
\dot{q} &= p,\\
\dot{p} &= -\sum_{\alpha\in\Delta'}\frac{\cot\alpha(q)}{\sin^2\alpha(q)} s_\alpha s_{-\alpha} H_\alpha,\\
\dot{s} &= [\, s, \mathcal{M}\,]
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M} = &-\sum_{\alpha\in<\pi'>}\left(\frac{1}{\sin^2\alpha(q)} - \frac{1}{3}\right)s_\alpha e_\alpha - \frac{5}{3}\sum_{\alpha\in\Delta\setminus<\pi'>} s_\alpha e_\alpha\\
&+\sum_{i,j} C_{ji}\sum_{\substack{\alpha\in<\pi'>-\pi'\\ \alpha_j-\alpha\in\Delta}} N_{\alpha,\alpha_j-\alpha}\left(\frac{1}{\sin^2\alpha(q)} - \frac{1}{3}\right)s_\alpha s_{\alpha_j-\alpha}h_{\alpha_i}\\
&+\frac{5}{3}\sum_{i,j} C_{ji}\sum_{\substack{\alpha\in\Delta\setminus<\pi'>\\ \alpha_j-\alpha\in\Delta}} N_{\alpha,\alpha_j-\alpha}s_\alpha s_{\alpha_j-\alpha}h_{\alpha_i}.
\end{aligned}$$

(Here the notation $N_{\alpha,\beta}$ is as in Proposition 3.1.4.)

Proposition 4.1.4. *The classical dynamical r-matrix R associated with the trigonometric dynamical r-matrix with spectral parameter in (4.1.1) is given by*

$$\begin{aligned}
(R(q)X)(z) = &\frac{1}{2}X(z) + \sum_{k=0}^{\infty}\frac{c^{(k)}(-z)}{k!}X_{-(k+1)} - \sum_{\alpha\in<\pi'>} c(\alpha(q))(X_{-1})_\alpha e_\alpha\\
&+ i\sum_{\alpha\in\overline{\pi}'^{+}}(X_{-1})_\alpha e_\alpha - i\sum_{\alpha\in\overline{\pi}'^{-}}(X_{-1})_\alpha e_\alpha.
\end{aligned}\tag{4.1.7}$$

Proof. The formula follows from (2.3.1) and (4.1.1) by a direct calculation where we have used the formula $\frac{d^k}{dw^k}\Big|_{w=0}\phi_\alpha(q,z-w) = c^{(k)}(-z),\ k\geq 1.$ $\square$

Corollary 4.1.5. *On $J^{-1}(0)$, we have*

$$\begin{aligned}
&(R(q)M(q,p,\xi))(z)\\
&= \frac{1}{2}M(q,p,\xi)(z) - c(z)p + \sum_{\alpha\in\Delta\setminus<\pi'>}\phi_\alpha(q,z)c(z)\xi_\alpha e_\alpha\\
&\quad+\sum_{\alpha\in<\pi'>}\phi_\alpha(q,z)(c(\alpha(q)) + c(z) - c(\alpha(q)+z))\xi_\alpha e_\alpha
\end{aligned}\tag{4.1.8}$$

where $M(q,p,\xi)(z) = L(q,p,\xi)(z)/z$.

Proof. The formula in (4.1.8) follows from (4.1.7) by algebra on using the following expansion in a deleted neighborhood of 0:

$$M(q,p,\xi)(z) = \frac{\xi}{z^2} + \frac{1}{z} M(q,p,\xi)_{-1} + O(1),$$

where

$$\begin{aligned} &M(q,p,\xi)_{-1} \\ &= p + \sum_{\alpha \in <\pi'>} c(\alpha(q))\xi_\alpha e_\alpha - i \sum_{\alpha \in \overline{\pi}'^+} \xi_\alpha e_\alpha + i \sum_{\alpha \in \overline{\pi}'^-} \xi_\alpha e_\alpha. \end{aligned}$$

□

Our next lemma is obvious from (4.1.7) and the expansion $c^{(k)}(-z) = -k! z^{-(k+1)} + O(1), k \geq 0$, in a deleted neighborhood of 0.

Lemma 4.1.6. *For* $X \in L\mathfrak{g}$, $R^+(q)X \in L\mathfrak{g}$.

Lemma 4.1.7. *(a) For* $X \in L\mathfrak{g}$, $R^-(q)X$ *has singularities at the points of the rank one lattice* $\pi\mathbb{Z}$*, and is holomorphic in* $\mathbb{C} \setminus \pi\mathbb{Z}$*. Moreover,* $R^-(q)X$ *is simply-periodic with period* π.

(b) The principal part of $R^-(q)X$ *at* $z = 0$ *is* $-(\Pi_- X)(z)$.

(c) $R^-(q)X$ *is bounded as* $z \to \infty$ *in a period strip with*

$$\lim_{y\to\infty} (R^-(q)X)(x+iy) = i\Pi_{\mathfrak{h}} X_{-1} + \sum_{\alpha \in <\pi'>} (i - c(\alpha(q)))(X_{-1})_\alpha e_\alpha + 2i \sum_{\alpha \in \overline{\pi}'^+} (X_{-1})_\alpha e_\alpha,$$

$$\lim_{y\to\infty} (R^-(q)X)(x-iy) = -i\Pi_{\mathfrak{h}} X_{-1} - \sum_{\alpha \in <\pi'>} (i + c(\alpha(q)))(X_{-1})_\alpha e_\alpha - 2i \sum_{\alpha \in \overline{\pi}'^-} (X_{-1})_\alpha e_\alpha.$$

Proof. (a) Clearly, $c^{(k)}(-z)$ are periodic with period π and meromorphic in $\mathbb{C}$ with poles at the points of the rank one lattice $\pi\mathbb{Z}$. Therefore, the assertion follows.
(b) This follows from the property that for $k \geq 0$, we have $c^{(k)}(-z) = -k! z^{-(k+1)} + O(1)$ in a deleted neighborhood of $z = 0$.
(c) First of all, note that $\lim_{y\to\pm\infty} \cot(x+iy) = \mp i$. On the other hand, it is easy to check that the derivatives of $\cot z$ always contain $\csc^2 z$ as a factor. Therefore, we have $\lim_{y\to\pm\infty} c^{(k)}(x+iy) = 0$ for $k \geq 1$. The formulas for $\lim_{y\to\infty}(R^-(q)X)(x \pm iy)$ are now obvious from (4.1.7). □

In order to describe the membership of the elements $(R^-(q)X)(\pm i\infty)$ in Lemma 4.1.7 (c) and for subsequent analysis, we need to introduce a number of Lie subalgebras of $\mathfrak{g}$ and their corresponding Lie groups. To begin with, let $\mathfrak{b}^- = \mathfrak{h} + \sum_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$

and $\mathfrak{b}^+ = \mathfrak{h} + \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$ be opposing Borel subalgebras of $\mathfrak{g}$. Then for each $\pi' \subset \pi$, we have the parabolic subalgebras

$$\mathfrak{p}^\pm_{\pi'} = \mathfrak{b}^\pm + \sum_{\alpha\in<\pi'>^\mp} \mathfrak{g}_\alpha. \tag{4.1.9}$$

Recall that $\mathfrak{p}^\pm_{\pi'}$ admit the following direct sum decomposition [Kn]

$$\mathfrak{p}^\pm_{\pi'} = \mathfrak{g}_{\pi'} + \mathfrak{n}^\pm_{\pi'} \tag{4.1.10}$$

where

$$\mathfrak{g}_{\pi'} = \mathfrak{h} + \sum_{\alpha\in<\pi'>} \mathfrak{g}_\alpha \tag{4.1.11}$$

is the Levi factor of $\mathfrak{p}^\pm_{\pi'}$, and

$$\mathfrak{n}^\pm_{\pi'} = \sum_{\alpha\in\overline{\pi}'^\pm} \mathfrak{g}_\alpha \tag{4.1.12}$$

are the nilpotent radicals. We shall denote by $\Pi^\pm_{\mathfrak{g}_{\pi'}}$ the projection maps onto $\mathfrak{g}_{\pi'}$ relative to the splitting $\mathfrak{p}^\pm_{\pi'} = \mathfrak{g}_{\pi'} + \mathfrak{n}^\pm_{\pi'}$. On the other hand, the connected and simply-connected Lie subgroups of G with corresponding Lie subalgebras $\mathfrak{p}^\pm_{\pi'}$, $\mathfrak{g}_{\pi'}$, and $\mathfrak{n}^\pm_{\pi'}$ will be denoted respectively by $P^\pm_{\pi'}$, $G_{\pi'}$, and $N^\pm_{\pi'}$ and we have $P^\pm_{\pi'} = N^\pm_{\pi'} G_{\pi'}$.

Thus it follows from Lemma 4.1.7 (c) that $(R^-(q)X)(\pm i\infty) \in \mathfrak{p}^\pm_{\pi'}$. The proof of the our next proposition is obvious.

Proposition 4.1.8. *(a)* $Im\mathcal{R}^+ = \bigcup_{q\in U}\{0_q\} \times L^+\mathfrak{g} \times \mathfrak{h}$.
(b) $\mathcal{I}^+ = \bigcup_{q\in U}\{0_q\} \times L^+\mathfrak{g} \times \{0\}$ *= adjoint bundle of* $Im\mathcal{R}^+$.

Remark 4.1.9. Indeed, in going through the proof of Proposition 4.1.8 (a) above, one can show that

$$\left\{\mathcal{R}^+(0_q, X, 0) \mid q \in U, X \in L\mathfrak{g}\right\} = \bigcup_{q\in U}\{0_q\} \times L^+\mathfrak{g} \times \mathfrak{h}.$$

Proposition 4.1.10. $Im\mathcal{R}^- = \mathcal{I}^- \bowtie \mathcal{Q}$, *where*

$$\mathcal{Q} = \left\{(0_q, -c(\cdot)Z, Z) \mid q \in U, Z \in \mathfrak{h}\,\right\} \tag{4.1.13}$$

is a Lie subalgebroid of $Im\mathcal{R}^-$ *and the ideal* $\mathcal{I}^-$ *coincides with the adjoint bundle of* $Im\mathcal{R}^-$ *and admits the following characterization:*

$$(0_q, X, 0) \in \mathcal{I}^-_q \text{ if and only if}$$

(a) X is holomorphic in $\mathbb{C} \setminus \pi\mathbb{Z}$ with singularities at the points of the rank one lattice $\pi\mathbb{Z}$,
(b) $X(z)$ is periodic with period π,
(c) $\Pi_{\mathfrak{h}} X_{-1} = 0$,
(d) X is bounded as $z \to \infty$ in a period strip with

$$\lim_{y\to\infty} X(x+iy) = \iota Z - \sum_{\alpha\in<\pi'>} (i - c(\alpha(q)))(X_{-1})_\alpha e_\alpha - 2i \sum_{\alpha\in\overline{\pi}'^+} (X_{-1})_\alpha e_\alpha,$$

$$\lim_{y\to\infty} X(x-iy) = \iota Z + \sum_{\alpha\in<\pi'>} (i + c(\alpha(q)))(X_{-1})_\alpha e_\alpha + 2i \sum_{\alpha\in\overline{\pi}'^-} (X_{-1})_\alpha e_\alpha,$$

for some $Z \in \mathfrak{h}$. Consequently, $X(\pm i\infty) \in \mathfrak{p}^{\pm}_{\pi'}$ and

$$\Pi^-_{\mathfrak{g}_{\pi'}} X(-i\infty) = Ad_{e^{2iq}} \Pi^+_{\mathfrak{g}_{\pi'}} X(i\infty). \tag{4.1.14}$$

Proof. From the definition of $\mathcal{I}^-$, we have

$$\begin{aligned}
&(0_q, X, 0) \in \mathcal{I}^- \\
\Longleftrightarrow\ & \mathcal{R}^+(0_q, X, Z) = 0 \text{ for some } Z \in \mathfrak{h} \\
\Longleftrightarrow\ & \Pi_{\mathfrak{h}} X_{-1} = 0,\ -\iota Z + R^+(q)X = 0 \text{ for some } Z \in \mathfrak{h} \\
\Longleftrightarrow\ & \Pi_{\mathfrak{h}} X_{-1} = 0,\ X(z) = \iota Z - (R^-(q)X + c(\cdot)\Pi_{\mathfrak{h}} X_{-1})(z) \text{ for some } Z \in \mathfrak{h}.
\end{aligned}$$

Therefore, by Lemma 4.1.7 above and the relation $c(\alpha(q))+i = e^{2i\alpha(q)}(c(\alpha(q))-i)$, we conclude that X satisfies the properties in (a)-(d). Conversely, suppose $X \in L\mathfrak{g}$ satisfies the properties in (a)-(d). Consider

$$D(z) = X(z) + (R^-(q)X)(z).$$

Then by the properties of X and Lemma 4.1.7, $D(z+\pi) = D(z)$ and the principal part of D at $z = 0$ is zero. Therefore, D extends to a holomorphic map from $\mathbb{C}$ to $\mathfrak{g}$. Moreover, D is bounded as $z \to \infty$ in the period strip and $\lim_{y\to\infty} D(x \pm iy) = \iota Z$. Write $D(z) = \sum_j d_j(z)x_j + \sum_{\alpha\in\Delta} d_\alpha(z)e_\alpha$. Then d_j and d_α are entire functions for $1 \le j \le N, \alpha \in \Delta$ and are periodic with period π. Therefore, when we combine this with the boundedness of d_j and d_α as $z \to \infty$ in the period strip, we conclude that $d_j(z) = d_j(=\text{constant})$ for each j and $d_\alpha(z) = d_\alpha(=\text{constant})$ for each α. But now it follows from $\lim_{y\to\infty} D(x \pm iy) = \iota Z$ that we must have $D(z) = \sum_j d_j x_j = \iota Z$. Consequently, $X = \iota Z - R^-(q)X$ and this in turn implies that $-\iota Z + R^+(q)X = 0$. As $\Pi_{\mathfrak{h}} X_{-1} = 0$, we have $(0_q, X, 0) \in \mathcal{I}^-_q$, as was to be proved. The proof of the assertion $Im\mathcal{R}^- = \mathcal{I}^- \bowtie \mathcal{Q}$ is similar to the one of Proposition 5.1.9 and so we will omit the details. □

4.2. Solution of the integrable trigonometric spin Calogero-Moser systems.

In principle, we have to solve the factorization problem

$$exp\{\, t(0,0,M(q^0,p^0,\xi^0))\}(q^0) = \gamma_+(t)\,\gamma_-(t)^{-1} \tag{4.2.1}$$

for $(\gamma_+(t),\gamma_-(t)) = ((q^0,k_+(t),q(t)),(q^0,k_-(t),q(t))) \in Im(\mathcal{R}^+,\mathcal{R}^-)$ satisfying the condition

$$\left(T_{\gamma_+(t)}\boldsymbol{l}_{\gamma_+(t)^{-1}}\dot{\gamma}_+(t), T_{\gamma_-(t)}\boldsymbol{l}_{\gamma_-(t)^{-1}}\dot{\gamma}_-(t)\right) \in (\mathcal{R}^+,\mathcal{R}^-)(\{q(t)\}\times\{0\}\times L\mathfrak{g}), \tag{4.2.2}$$

where $(q^0,p^0,\xi^0) \in J^{-1}(0) = TU \times (\mathcal{U}\cap\mathfrak{h}^\perp)$ is the initial value of (q,p,ξ) and $M(q^0,p^0,\xi^0)(z) = L(q^0,p^0,\xi^0)(z)/z$. (We shall denote $k_\pm(t)(z)$ by $k_\pm(z,t)$.) However, as we shall see in the next two propositions and their corollary, it actually suffices to solve for $q(t)$, $k_+(0,t)$ and $k_-(\pm i\infty,t)$ and we will find the factorization problems for these quantities from (4.2.1) and (4.2.2) above. In what follows, we shall denote by $(q(t),p(t),\xi(t))$ the Hamiltonian flow on $J^{-1}(0)$ generated by $\mathcal{H}$ with initial condition $(q(0),p(0),\xi(0)) = (q^0,p^0,\xi^0)$.

Proposition 4.2.1. *With the notations introduced above,*
(a) $L(q(t),p(t),\xi(t))(\pm i\infty)$ exist. Explicitly,

$$\begin{aligned}
&L(q(t),p(t),\xi(t))(\pm i\infty)\\
&= L(q(t),p(t),\xi(t))(\pi/2) \mp i\xi(t)\\
&= p(t) + \sum_{\alpha\in<\pi'>} (c(\alpha(q(t))) \mp i)\xi_\alpha(t)e_\alpha \mp 2i \sum_{\alpha\in\pi'^{\pm}} \xi_\alpha(t)e_\alpha
\end{aligned} \tag{4.2.3}$$

and therefore $L(q(t),p(t),\xi(t))(\pm i\infty) \in \mathfrak{p}^\pm_{\pi'}$.
(b) $L(q(t),p(t),\xi(t))(\pm i\infty)$ satisfy the Lax equations

$$\begin{aligned}
&\frac{d}{dt}L(q(t),p(t),\xi(t))(\pm i\infty)\\
&= \left[L(q(t),p(t),\xi(t))(\pm i\infty),\, -\sum_{\alpha\in<\pi'>} \csc^2(\alpha(q(t)))\xi_\alpha(t)e_\alpha\right].
\end{aligned} \tag{4.2.4}$$

Proof. (a) The existence of $L(q(t),p(t),\xi(t))(\pm i\infty)$ and their explicit formulas are obtained from (4.1.4) by noting that $\lim_{y\to\pm\infty}\cot(iy) = \mp i$.
(b) According to Proposition 2.3.5, we have

$$\dot{L}(q(t),p(t),\xi(t))(z) = [\,L(q(t),p(t),\xi(t))(z),(R^-(q(t))M(q(t),p(t),\xi(t)))(z)\,]$$

where

$$
\begin{aligned}
&(R^-(q(t))M(q(t),p(t),\xi(t)))(z)\\
= &- c(z)p(t) + \sum_{\alpha\in\Delta\setminus<\pi'>} \phi_\alpha(q(t),z)c(z)\xi_\alpha(t)e_\alpha\\
&+ \sum_{\alpha\in<\pi'>} \phi_\alpha(q(t),z)(c(\alpha(q(t))) + c(z) - c(\alpha(q(t)) + z))\xi_\alpha(t)e_\alpha
\end{aligned}
$$

by (4.1.8). Now, it is easy to see from (4.1.3) that

$$
\phi_\alpha(q(t),i\infty) = \begin{cases} -(-i+\cot\alpha(q(t))), & \alpha\in<\pi'> \\ 2i, & \alpha\in\overline{\pi}'^{+} \\ 0, & \alpha\in\overline{\pi}'^{-} \end{cases}
$$

whereas

$$
\phi_\alpha(q(t),-i\infty) = \begin{cases} -(i+\cot\alpha(q(t))), & \alpha\in<\pi'> \\ 0, & \alpha\in\overline{\pi}'^{+} \\ -2i, & \alpha\in\overline{\pi}'^{-}. \end{cases}
$$

Therefore, upon taking the limit as $z = iy \to \pm i\infty$ in the above expression for $(R^-(q(t))M(q(t),p(t),\xi(t)))(z)$, we find that

$$
\begin{aligned}
&(R^-(q(t))M(q(t),p(t),\xi(t)))(\pm i\infty)\\
= &\pm iL(q(t),p(t),\xi(t))(\pm i\infty) - \sum_{\alpha\in<\pi'>} \csc^2(\alpha(q(t)))\xi_\alpha(t)e_\alpha
\end{aligned}
$$

from which the assertion follows. □

Remark 4.2.2. Although $L(q,p,\xi)(\pm i\infty)$ exist and satisfy Lax equations, however, they are deficient in the sense that they do not provide enough conserved quantities for complete integrability. In order to establish Liouville integrability, we must use the Lax operator with spectral parameter $L(q,p,\xi)(z)$.

We next spell out some of the consequences of the condition in (4.2.2) which will clarify the relation between the term $-\sum_{\alpha\in<\pi'>}\csc^2(\alpha(q(t)))\xi_\alpha(t)e_\alpha$ which appears in the Lax equations above for $L(q(t),p(t),\xi(t))(\pm i\infty)$ and the factors $k_\pm(z,t)$.

Proposition 4.2.3. *(a) $k_+(0,t)\in G_{\pi'}$ and satisfies the equation*

$$
T_{k_+(0,t)}l_{k_+(0,t)^{-1}}\dot{k}_+(0,t) = -\sum_{\alpha\in<\pi'>} \csc^2(\alpha(q(t)))\xi_\alpha(t)e_\alpha.
$$

(b) $k_-(\pm i\infty, t) \in P^{\pm}_{\pi'}$ *and satisfy the equations*

$$\begin{aligned} & T_{k_-(\pm i\infty,t)} l_{k_-(\pm i\infty,t)^{-1}} \dot{k}_-(\pm i\infty, t) \\ & = \pm i L(q(t), p(t), \xi(t))(\pm i\infty) - \sum_{\alpha \in <\pi'>} \csc^2(\alpha(q(t)))\xi_\alpha(t) e_\alpha. \end{aligned}$$

Proof. (a) It follows from (4.2.1) and (4.2.2) that (see the proof of Theorem 2.2.2 in [L2])

$$T_{\gamma_+(t)} \boldsymbol{l}_{\gamma_+(t)^{-1}} \dot{\gamma}_+(t) = \mathcal{R}^+(q(t), 0, M(q(t), p(t), \xi(t))).$$

Consequently, we have

$$\begin{aligned} & T_{k_+(z,t)} l_{k_+(z,t)^{-1}} \dot{k}_+(z, t) \\ = & (R^+(q(t)) M(q(t), p(t), \xi(t)))(z) \\ = & M(q(t), p(t), \xi(t))(z) - c(z)p(t) + \sum_{\alpha \in \Delta \setminus <\pi'>} \phi_\alpha(q(t), z) c(z) \xi_\alpha(t) e_\alpha \\ & + \sum_{\alpha \in <\pi'>} \phi_\alpha(q(t), z)(c(\alpha(q(t))) + c(z) - c(\alpha(q(t)) + z))\xi_\alpha(t) e_\alpha. \qquad (*) \end{aligned}$$

Since $\cot z = \frac{1}{z} + O(z^3)$ in a deleted neighborhood of 0, the z^0 term in the Laurent series expansion about 0 of $M(q(t), p(t), \xi(t))(z)$, $c(z)p(t)$ and $\phi_\alpha(q(t), z)c(z)$ for $\alpha \in \Delta \setminus < \pi' >$ is equal to zero in each case. On the other hand, for $\alpha \in < \pi' >$, the z^0 term in the Laurent series expansion of $\phi_\alpha(q(t), z)(c(\alpha(q(t))) + c(z) - c(\alpha(q(t)) + z))$ about 0 is $-\csc^2(\alpha(q(t)))$. The formula for $T_{k_+(0,t)} l_{k_+(0,t)^{-1}} \dot{k}_+(0, t)$ thus follows when we let $z \to 0$ in (*) above.
(b) It also follows from (4.2.1) and (4.2.2) that (see the proof of Theorem 2.2.2 of [L2])

$$T_{\gamma_-(t)} \boldsymbol{l}_{\gamma_-(t)^{-1}} \dot{\gamma}_-(t) = \mathcal{R}^-(q(t), 0, M(q(t), p(t), \xi(t)))$$

and hence

$$\begin{aligned} & T_{k_-(z,t)} l_{k_-(z,t)^{-1}} \dot{k}_-(z, t) \\ = & (R^-(q(t)) M(q(t), p(t), \xi(t)))(z). \end{aligned}$$

The formulas for $T_{k_-(\pm i\infty,t)} l_{k_-(\pm i\infty,t)^{-1}} \dot{k}_-(\pm i\infty, t)$ then follow from the proof of Proposition 4.2.1 (b). Finally the assertion that $k_-(\pm i\infty, t) \in P^{\pm}_{\pi'}$ is a consequence of these formulas and Proposition 4.2.1 (a). □

Corollary 4.2.4. *In terms of* $k_+(0, t)$, *we have*

$$L(q(t), p(t), \xi(t))(\pm i\infty) = Ad_{k_+(0,t)^{-1}} L(q^0, p^0, \xi^0)(\pm i\infty).$$

Consequently,

$$L(q(t),p(t),\xi(t))(z) = Ad_{k_+(0,t)^{-1}} L(q^0,p^0,\xi^0)(z).$$

Proof. By using Proposition 4.2.3 (a) and Proposition 4.2.1 (b), we can check that $Ad_{k_+(0,t)} L(q(t),p(t),\xi(t))(\pm i\infty)$ are constants, hence

$$Ad_{k_+(0,t)} L(q(t),p(t),\xi(t))(\pm i\infty) = L(q^0,p^0,\xi^0)(\pm i\infty).$$

Now it is clear from (4.2.3) that

$$2L(q(t),p(t),\xi(t))(\pi/2) = L(q(t),p(t),\xi(t))(i\infty) + L(q(t),p(t),\xi(t))(-i\infty)$$

and

$$-2i\xi(t) = L(q(t),p(t),\xi(t))(i\infty) - L(q(t),p(t),\xi(t))(-i\infty).$$

As

$$L(q(t),p(t),\xi(t))(z) = L(q(t),p(t),\xi(t))(\pi/2) + c(z)\xi(t),$$

the second assertion is a consequence of the first one by virtue of the above relations. □

Combining the formulas in Proposition 4.2.3 (a) and (b), and the fact that

$$L(q(t),p(t),\xi(t))(\pm i\infty) = Ad_{k_-(\pm i\infty,t)^{-1}} L(q^0,p^0,\xi^0)(\pm i\infty), \tag{4.2.5}$$

we obtain the following factorization problems on $P^{\pm}_{\pi'}$:

$$e^{itL(q^0,p^0,\xi^0)(i\infty)} = k_-(i\infty,t)k_+(0,t)^{-1}, \tag{4.2.6}$$

$$e^{-itL(q^0,p^0,\xi^0)(-i\infty)} = k_-(-i\infty,t)k_+(0,t)^{-1} \tag{4.2.7}$$

where $k_+(0,t)$ and $k_-(\pm i\infty)$ are to be determined. The nature of these factorization problems are of course quite different from that of those in the well-known group-theoretic scheme for constant r-matrices.(Compare, for example, the factorization problems in [RSTS],[STS], [DLT] with our solution of (4.2.6), (4.2.7) below.)

We shall use the following notation: for $g^{\pm} \in P^{\pm}_{\pi'}$, $\boldsymbol{\nu}^{\pm}(g^{\pm}) \in N^{\pm}_{\pi'}$, $\boldsymbol{\lambda}^{\pm}(g^{\pm}) \in G_{\pi'}$ will denote the factors in the unique factorization $g^{\pm} = \boldsymbol{\nu}^{\pm}(g^{\pm})\boldsymbol{\lambda}^{\pm}(g^{\pm})$. In order to solve (4.2.6) and (4.2.7), note that

$$(q^0,k_-(t),q(t)) = (q^0,\widehat{k}_-(t),q^0)(q^0,e^{c(\cdot)(q^0-q(t))},q(t)) \tag{4.2.8}$$

by the global version of Proposition 4.1.10 where $(q^0, \widehat{k}_-(t), q^0)$ is in the Lie group bundle integrating $\mathcal{I}^-$ and the second factor $(q^0, e^{c(\cdot)(q^0-q(t))}, q(t))$ is in the Lie groupoid integrating $\mathcal{Q}$. Consequently, the factorization problems on $P^{\pm}_{\pi'}$ in (4.2.6) and (4.2.7) can be recast in the form

$$e^{itL(q^0,p^0,\xi^0)(i\infty)} = \boldsymbol{\nu}^+(\widehat{k}_-(i\infty,t))\boldsymbol{\lambda}^+(\widehat{k}_-(i\infty,t))e^{-i(q^0-q(t))}k_+(0,t)^{-1}, \tag{4.2.9}$$

$$e^{-itL(q^0,p^0,\xi^0)(-i\infty)} = \boldsymbol{\nu}^-(\widehat{k}_-(-i\infty,t))\boldsymbol{\lambda}^-(\widehat{k}_-(-i\infty,t))e^{i(q^0-q(t))}k_+(0,t)^{-1}. \tag{4.2.10}$$

Now, from the fact that $e^{itL(q^0,p^0,\xi^0)(i\infty)} \in P^+_{\pi'}$, we can find unique $n_+(t) \in N^+_{\pi'}$, $g_+(t) \in G_{\pi'}$ satisfying $n_+(0) = g_+(0) = 1$ such that

$$e^{itL(q^0,p^0,\xi^0)(i\infty)} = n_+(t)g_+(t). \tag{4.2.11}$$

Similarly, we can find unique $n_-(t) \in N^-_{\pi'}$, $g_-(t) \in G_{\pi'}$ satisfying $n_-(0) = g_-(0) = 1$ such that

$$e^{-itL(q^0,p^0,\xi^0)(i\infty)} = n_-(t)g_-(t). \tag{4.2.12}$$

By comparing (4.2.9) (resp. (4.2.10))with (4.2.11) (resp. (4.2.12)), we obtain

$$\boldsymbol{\nu}^+(\widehat{k}_-(i\infty,t)) = n_+(t),\ \boldsymbol{\nu}^-(\widehat{k}_-(-i\infty,t)) = n_-(t). \tag{4.2.13}$$

Hence the factorization problems reduce to

$$g_+(t) = \boldsymbol{\lambda}^+(\widehat{k}_-(i\infty,t))e^{-i(q^0-q(t))}k_+(0,t)^{-1}, \tag{4.2.14}$$

$$g_-(t) = \boldsymbol{\lambda}^-(\widehat{k}_-(-i\infty,t))e^{i(q^0-q(t))}k_+(0,t)^{-1}. \tag{4.2.15}$$

But from the global version of (4.1.14), we have

$$\boldsymbol{\lambda}^-(\widehat{k}_-(-i\infty,t)) = e^{2iq^0}\boldsymbol{\lambda}^+(\widehat{k}_-(i\infty,t))e^{-2iq^0}. \tag{4.2.16}$$

Substitute this into (4.2.15) above, we find

$$e^{-2iq^0}g_-(t) = \boldsymbol{\lambda}^+(\widehat{k}_-(i\infty,t))e^{-i(q^0+q(t))}k_+(0,t)^{-1}. \tag{4.2.17}$$

Consequently, when we eliminate $\boldsymbol{\lambda}^+(\widehat{k}_-(i\infty,t))$ from (4.2.14) and (4.2.17), we obtain the following factorization problem on $G_{\pi'}$:

$$g_-(t)^{-1}e^{2iq^0}g_+(t) = k_+(0,t)e^{2iq(t)}k_+(0,t)^{-1}. \tag{4.2.18}$$

But $G_{\pi'}$ is a reductive Lie group, hence we can find (for at least small values of t) $x(t) \in G_{\pi'}$ (unique to transformations $x(t) \to x(t)\delta(t)$ where $\delta(t) \in H$) and unique $d(t) \in H$ such that

$$g_-(t)^{-1} e^{2iq^0} g_+(t) = x(t)d(t)x(t)^{-1} \tag{4.2.19}$$

with $x(0) = 1, d(0) = e^{2iq^0}$. This determines $q(t)$ via the formula

$$q(t) = \frac{1}{2i} log\, d(t). \tag{4.2.20}$$

On the other hand, let us fix one such $x(t)$. We shall seek $k_+(0,t)$ in the form

$$k_+(0,t) = x(t)h(t), \quad h(t) \in H. \tag{4.2.21}$$

To determine $h(t)$, we shall impose the following condition (which is a corollary of Proposition 4.2.3 (a)):

$$\Pi_{\mathfrak{h}} T_{k_+(0,t)} l_{k_+(0,t)^{-1}} \dot{k}_+(0,t) = 0 \tag{4.2.22}$$

where $\Pi_{\mathfrak{h}}$ is the projection map to $\mathfrak{h}$ relative to the direct sum decomposition $\mathfrak{g}_{\pi'} = \mathfrak{h} + \sum_{\alpha \in <\pi'>} \mathfrak{g}_\alpha$. Substitute (4.2.21) into (4.2.22), we see that $h(t)$ satisfies the equation

$$\dot{h}(t) = -T_e l_{h(t)} (\Pi_{\mathfrak{h}} T_{x(t)} l_{x(t)^{-1}} \dot{x}(t)) \tag{4.2.23}$$

with $h(0) = 1$. Solving the equation explicitly, we find that

$$h(t) = exp\left\{ -\int_0^t \Pi_{\mathfrak{h}} (T_{x(\tau)} l_{x(\tau)^{-1}} \dot{x}(\tau))\, d\tau \right\}. \tag{4.2.24}$$

Theorem 4.2.5. *Let* $(q^0, p^0, \xi^0) \in J^{-1}(0) = TU \times (\mathcal{U} \cap \mathfrak{h}^\perp)$. *Then the Hamiltonian flow on* $J^{-1}(0)$ *generated by*

$$\begin{aligned}\mathcal{H}(q,p,\xi) =& \frac{1}{2}\sum_i p_i^2 - \frac{1}{2}\sum_{\alpha \in <\pi'>} \left(\frac{1}{\sin^2 \alpha(q)} - \frac{1}{3} \right) \xi_\alpha \xi_{-\alpha} - \frac{5}{6} \sum_{\alpha \in \Delta \setminus <\pi'>} \xi_\alpha \xi_{-\alpha} \\ & - \frac{1}{3} \sum_i \xi_i^2.\end{aligned}$$

with initial condition $(q(0), p(0), \xi(0)) = (q^0, p^0, \xi^0)$ *is given by*

$$\begin{aligned} q(t) =& \frac{1}{2i} log\, d(t), \\ \xi(t) =& Ad_{k_+(0,t)^{-1}} \xi^0, \\ p(t) =& Ad_{k_+(0,t)^{-1}} L(q^0, p^0, \xi^0)(\pm i\infty) - \sum_{\alpha \in <\pi'>} (c(\alpha(q(t))) \mp i) \xi_\alpha(t) e_\alpha \\ & \pm 2i \sum_{\alpha \in \overline{\pi}'^{\pm}} \xi_\alpha(t) e_\alpha \end{aligned} \tag{4.2.25}$$

where $d(t)$ and $k_+(0,t)$ are constructed from the above procedure.

Proof. The formula for $\xi(t)$ is a consequence of Corollary 4.2.4 and the relation

$$-2i\xi(t) = L(q(t),p(t),\xi(t))(i\infty) - L(q(t),p(t),\xi(t))(-i\infty).$$

On the other hand, the formula for $p(t)$ follows by equating the two different expressions for $L(q(t),p(t),\xi(t))(\pm i\infty)$ in (4.2.3) and in Corollary 4.2.4. $\square$

By Poisson reduction, we can now write down the solution of the associated integrable model on $TU \times \mathfrak{g}_{red}$ with Hamiltonian $\mathcal{H}_0$ whose equations of motion are given in Proposition 4.1.3, as in Corollary 3.2.2.

Corollary 4.2.6. *Let $(q^0,p^0,s^0) \in TU \times \mathfrak{g}_{red}$ and suppose $s^0 = Ad_{g(\xi^0)^{-1}}\xi^0$ where $\xi^0 \in \mathcal{U} \cap \mathfrak{h}^\perp$. Then the Hamiltonian flow generated by $\mathcal{H}_0$ with initial condition $(q(0),p(0),s(0)) = (q^0,p^0,s^0)$ is given by*

$$\begin{aligned}
q(t) =& \; d(t),\\
s(t) =& \; Ad_{\left(\widetilde{k}_+(0,t)\, g\left(Ad_{\widetilde{k}_+(0,t)^{-1}} s^o\right)\right)^{-1}} s^0,\\
p(t) =& \; Ad_{\left(\widetilde{k}_+(0,t)\, g\left(Ad_{\widetilde{k}_+(0,t)^{-1}} s^o\right)\right)^{-1}} L(q^0,p^0,s^0)(\pm i\infty)\\
&- \sum_{\alpha \in <\pi'>} (c(\alpha(q(t))) \mp i) s_\alpha(t) e_\alpha \pm 2i \sum_{\alpha \in \overline{\pi}'^{\pm}} s_\alpha(t) e_\alpha
\end{aligned} \tag{4.2.26}$$

where $\widetilde{k}_+(0,t) = g(\xi^0)^{-1} k_+(0,t) g(\xi^0)$ and $k_+(0,t)$, $d(t)$ are as in Theorem 4.2.5.

Remark 4.2.7. The reader should contrast the factorization problems in this section with the ones in [L2]. Although the Hamiltonians are rather similar (we can transform the hyperbolic spin CM systems in [L2] to trigonometric ones), however, the factorization problems involved are quite different.

5. The elliptic spin Calogero-Moser systems.

5.1. Lax operators, Hamiltonian equations and the Lie subalgebroids.

In this section, $\wp(z)$ is the Weierstrass $\wp$-function with periods $2\omega_1, 2\omega_2 \in \mathbb{C}$, and $\sigma(z)$, $\zeta(z)$ are the related Weierstrass sigma-function and zeta-function.

We consider the following elliptic dynamical r-matrix with spectral parameter, given by

$$r(q,z) = \zeta(z) \sum_i x_i \otimes x_i - \sum_{\alpha \in \Delta} l(\alpha(q), z) e_\alpha \otimes e_{-\alpha} \tag{5.1.1}$$

where

$$l(w,z) = -\frac{\sigma(w+z)}{\sigma(w)\sigma(z)}. \tag{5.1.2}$$

Then the associated spin Calogero-Moser system on $TU \times \mathfrak{g}$ is called the elliptic spin Calogero-Moser system. Explicitly, the Hamiltonian is of the form

$$\mathcal{H}(q,p,\xi) = \frac{1}{2}\sum_i p_i^2 - \frac{1}{2}\sum_{\alpha\in\Delta} \wp(\alpha(q))\xi_\alpha\xi_{-\alpha} \tag{5.1.3}$$

and the Lax operator is given by

$$L(q,p,\xi)(z) = p + \zeta(z)\sum_i \xi_i x_i - \sum_{\alpha\in\Delta} l(\alpha(q),z)\xi_\alpha e_\alpha. \tag{5.1.4}$$

Our next result gives the Hamiltonian equations of motion generated by $\mathcal{H}$. Using the same method of calculation as in the proof of Proposition 3.1.4, we can also compute the corresponding equations generated by its reduction

$$\mathcal{H}_0(q,p,s) = \frac{1}{2}\sum_i p_i^2 - \frac{1}{2}\sum_{\alpha\in\Delta} \wp(\alpha(q)) s_\alpha s_{-\alpha} \tag{5.1.5}$$

on $TU \times \mathfrak{g}_{red}$.

Proposition 5.1.1. *The Hamiltonian equations of motion generated by $\mathcal{H}$ on $TU\times \mathfrak{g}$ are given by*

$$\begin{aligned} \dot{q} &= p, \\ \dot{p} &= \frac{1}{2}\sum_{\alpha\in\Delta} \wp'(\alpha(q))\xi_\alpha\xi_{-\alpha}H_\alpha, \\ \dot{\xi} &= \left[\xi, -\sum_{\alpha\in\Delta}\wp(\alpha(q))\xi_\alpha e_\alpha\right]. \end{aligned} \tag{5.1.6}$$

Proposition 5.1.2. *The Hamiltonian equations of motion generated by $\mathcal{H}_0$ on the reduced Poisson manifold $TU \times \mathfrak{g}_{red}$ are given by*

$$\begin{aligned} \dot{q} &= p, \\ \dot{p} &= \frac{1}{2}\sum_{\alpha\in\Delta} \wp'(\alpha(q)) s_\alpha s_{-\alpha} H_\alpha, \\ \dot{s} &= [\, s, \mathcal{M}\,] \end{aligned}$$

where

$$\mathcal{M} = -\sum_{\alpha\in\Delta}\wp(\alpha(q))s_\alpha e_\alpha + \sum_{i,j} C_{ji} \sum_{\substack{\alpha\in\Delta \\ \alpha_j-\alpha\in\Delta}} N_{\alpha,\alpha_j-\alpha}\,\wp(\alpha(q)) s_\alpha s_{\alpha_j-\alpha} h_{\alpha_i}.$$

(Here the notation $N_{\alpha,\beta}$ is as in Proposition 3.1.4.)

Proposition 5.1.3. *The classical dynamical r-matrix R associated with the elliptic dynamical r-matrix with spectral parameter in (5.1.1) is given by*

$$\begin{aligned}(R(q)X)(z) = &\frac{1}{2}X(z) + \sum_{k=0}^{\infty}\frac{\zeta^{(k)}(-z)}{k!}\,\Pi_{\mathfrak{h}}X_{-(k+1)} \\ &+\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\alpha\in\Delta}\frac{d^k}{dw^k}\Big|_{w=0} l(\alpha(q), z-w)(X_{-(k+1)})_{\alpha}e_{\alpha}.\end{aligned}\tag{5.1.7}$$

Proof. Here, we have used the formula $l(-w,z) = -l(w,-z)$. Otherwise, the proof is similar to that of Proposition 4.1.4. □

Corollary 5.1.4. *On $J^{-1}(0)$, we have*

$$\begin{aligned}&(R(q)M(q,p,\xi))(z)\\ =&\frac{1}{2}M(q,p,\xi)(z) - \zeta(z)p + \sum_{\alpha\in\Delta} l(\alpha(q),z)(\zeta(\alpha(q)) + \zeta(z) - \zeta(\alpha(q)+z))\xi_{\alpha}e_{\alpha}\end{aligned}\tag{5.1.8}$$

where $M(q,p,\xi)(z) = L(q,p,\xi)(z)/z$.

Proof. In a deleted neighborhood of 0, we have the expansion

$$M(q,p,\xi)(z) = \frac{\xi}{z^2} + \frac{1}{z}\left(p + \sum_{\alpha}\zeta(\alpha(q))\xi_{\alpha}e_{\alpha}\right) + O(1).$$

On the other hand, by direct differentiation, we find

$$\frac{d}{dw}\Big|_{w=0} l(\alpha(q), z-w) = -l(\alpha(q),z)(\zeta(\alpha(q)+z) - \zeta(z)).$$

Therefore, on using (5.1.7), we obtain the desired formula. □

Remark 5.1.5. Using (5.1.8), we can check that in this case, the equations in (5.1.6) can be recovered from the Lax equation $\dot{L}(q,p,\xi) = [L(q,p,\xi), R(q)M(q,p,\xi)]$ on $J^{-1}(0)$. The computation makes use of the following identities:
(i) $l(w,z)l(-w,z) = \wp(z) - \wp(w)$,
(ii) $l(x,z)l(y,z)[\zeta(x+z) - \zeta(x) - \zeta(y+z) + \zeta(y)] = l(x+y,z)[\wp(x) - \wp(y)]$,
(iii) $\zeta(x+y) - \zeta(x) - \zeta(y) = \frac{1}{2}\frac{\wp'(x)-\wp'(y)}{\wp(x)-\wp(y)}$.
We shall leave the details to the interested reader.

Our next lemma is a simple consequence of the fact that for $k \geq 0$, we have $\zeta^{(k)}(-z) = -k!z^{-(k+1)} + O(1)$, $\frac{d^k}{dw^k}\Big|_{w=0} l(\alpha(q), z-w) = -k!z^{-(k+1)} + O(1)$ in a deleted neighborhood of 0.

Lemma 5.1.6. *For $X \in L\mathfrak{g}$, $R^+(q)X \in L^+\mathfrak{g}$.*

Lemma 5.1.7. *For $X \in L\mathfrak{g}$, $R^-(q)X + \zeta(\cdot)\Pi_{\mathfrak{h}}X_{-1}$ has singularities at the points of the rank 2 lattice*

$$\Lambda = 2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z} \tag{5.1.9}$$

and is holomorphic in $\mathbb{C}\backslash\Lambda$. Moreover, the quasi-periodicity condition

$$(R^-(q)X + \zeta(\cdot)\Pi_{\mathfrak{h}}X_{-1})(z + 2\omega_i) = Ad_{e^{2\eta_i q}}(R^-(q)X + \zeta(\cdot)\Pi_{\mathfrak{h}}X_{-1})(z) \tag{5.1.10}$$

holds, where $\eta_i = \zeta(\omega_i)$, $i = 1, 2$.

Proof. From (5.1.7), we obtain

$$\begin{aligned}&(R^-(q)X + \zeta(\cdot)\Pi_{\mathfrak{h}}\, X_{-1})(z)\\ &= \sum_{k=1}^{\infty} \frac{\zeta^{(k)}(-z)}{k!}\,\Pi_{\mathfrak{h}}X_{-(k+1)} + \sum_{k=0}^{\infty}\sum_{\alpha\in\Delta}\frac{1}{k!}\frac{d^k}{dw^k}\Big|_{w=0} l(\alpha(q), z-w)(X_{-(k+1)})_\alpha e_\alpha\end{aligned}$$

from which it is clear that $R^-(q)X + \zeta(\cdot)\Pi_{\mathfrak{h}}\, X_{-1}$ is holomorphic in $\mathbb{C}\backslash\Lambda$ with singularities at the points of Λ. On the other hand, it is easy to check that

$$\frac{d^k}{dw^k}\Big|_{w=0} l(\alpha(q), z + 2\omega_i - w) = e^{2\eta_i\alpha(q)}\frac{d^k}{dw^k}\Big|_{w=0} l(\alpha(q), z-w), \quad k \geq 0.$$

Hence the second assertion follows. □

The proof of our next proposition is obvious.

Proposition 5.1.8. *(a) $Im\mathcal{R}^+ = \bigcup_{q\in U}\{0_q\} \times L^+\mathfrak{g} \times \mathfrak{h}$.*
(b) $\mathcal{I}^+ = \bigcup_{q\in U}\{0_q\} \times L^+\mathfrak{g} \times \{0\}$ = adjoint bundle of $Im\mathcal{R}^+$.

Indeed, we can show more, namely,

$$\left\{\mathcal{R}^+(0_q, X, 0) \mid q \in U, X \in L\mathfrak{g}\right\} = \bigcup_{q\in U}\{0_q\} \times L^+\mathfrak{g} \times \mathfrak{h}. \tag{5.1.11}$$

Proposition 5.1.9. *$Im\mathcal{R}^- = \mathcal{I}^- \bowtie \mathcal{Q}$, where*

$$\mathcal{Q} = \left\{(0_q, -\zeta(\cdot)Z, Z) \mid q \in U, Z \in \mathfrak{h}\,\right\} \tag{5.1.12}$$

is a Lie subalgebroid of $Im\mathcal{R}^-$ and the ideal $\mathcal{I}^-$ coincides with the adjoint bundle of $Im\mathcal{R}^-$ and admits the following characterization:

$$(0_q, X, 0) \in \mathcal{I}^-_q \text{ if and only if}$$

(a) X *is holomorphic in* $\mathbb{C}\backslash\Lambda$ *with singularities at the points of* Λ,
(b) $X(z+2\omega_i) = Ad_{e^{2\eta_i q}} X(z),\ i = 1,2,$
(c) $\Pi_{\mathfrak{h}} X_{-1} = 0$.

Proof. As in the proof of Proposition 4.1.10, we have

$$\begin{aligned} &(0_q, X, 0) \in \mathcal{I}^- \\ \Longleftrightarrow\ &\Pi_{\mathfrak{h}} X_{-1} = 0,\ X(z) = \iota Z - (R^-(q)X + \zeta(\cdot)\Pi_{\mathfrak{h}} X_{-1})(z) \text{ for some } Z \in \mathfrak{h}. \end{aligned}$$

Therefore, it follows from Lemma 5.1.7 that X satisfies the properties in (a)-(c). Conversely, suppose $X \in L\mathfrak{g}$ satisfies the properties in (a)-(c). Consider

$$D(z) = X(z) + (R^-(q)X)(z).$$

Then $D(z)$ satisfies the quasi-periodicity condition $D(z+2\omega_i) = Ad_{e^{2\eta_i q}} D(z), i = 1,2$ and the principal part of D at 0 is zero. Hence D extends to a holomorphic map from $\mathbb{C}$ to $\mathfrak{g}$. Write $D(z) = \sum_j d_j(z)x_j + \sum_{\alpha\in\Delta} d_\alpha(z)e_\alpha$. Then d_j and d_α are entire functions for $1 \le j \le N, \alpha \in \Delta$ and the quasi-periodicity condition implies that for $i = 1,2$, we have

$$d_j(z+2\omega_i) = d_j(z), \quad 1 \le j \le N, \tag{5.1.13}$$

$$d_\alpha(z+2\omega_i) = d_\alpha(z)e^{2\eta_i\alpha(q)}, \quad \alpha \in \Delta. \tag{5.1.14}$$

From (5.1.13) and Liouville's theorem, it follows that $d_j(z) = d_j(=\text{constant})$ for each j. On the other hand, observe that the meromorphic function $l(\alpha(q), z)$ satisfies the same quasi-periodicity condition as d_α, that is, $l(\alpha(q), z + 2\omega_i) = l(\alpha(q), z)e^{2\eta_i\alpha(q)}$. Hence we conclude from (5.1.14) and the above observation that $d_\alpha(z) = f_\alpha(z)l(\alpha(q), z)$, where f_α is an elliptic function. But the order of a non-constant elliptic function is never less than 2. Therefore, as $l(\alpha(q), z)$ has simple poles at the points of Λ, we must have $f_\alpha \equiv d_\alpha \equiv 0$ for each $\alpha \in \Delta$. Hence we have shown that $X = \iota Z - R^-(q)X$, where $Z = \sum_j d_j x_j$. This in turn implies that $-\iota Z + R^+(q)X = 0$ and so $(0_q, X, 0) \in \mathcal{I}^-$.

Next, we show $Im\mathcal{R}^- = \mathcal{I}^- \oplus \mathcal{Q}$. Consider an arbitrary element $\mathcal{R}^-(0_q, X, Z)$ in $Im\mathcal{R}^-$. Clearly, we have the decomposition

$$\begin{aligned} &\mathcal{R}^-(0_q, X, Z) \\ =\ &(0_q, -\iota Z + R^-(q)X + \zeta(\cdot)\Pi_{\mathfrak{h}} X_{-1}, 0) \\ &+ (0_q, -\zeta(\cdot)\Pi_{\mathfrak{h}} X_{-1}, \Pi_{\mathfrak{h}} X_{-1}) \end{aligned}$$

where the first term is in $\mathcal{I}^-$ (by Lemma 5.1.7 and the characterization of $\mathcal{I}^-$ which we established above) and the second term is in $\mathcal{Q}$. This shows that $Im\mathcal{R}^- \subset \mathcal{I}^- \oplus \mathcal{Q}$. Conversely, take an arbitrary element $(0_q, X, 0) + (0_q, -\zeta(\cdot)Z, Z) \in \mathcal{I}^- \oplus \mathcal{Q}$. From the definition of $\mathcal{I}^-$, we have $\mathcal{R}^+(0_q, X, Z') = 0$ for some $Z' \in \mathfrak{h}$. Let $Y = -X + \zeta(\cdot)Z$. Then

$$\begin{aligned}
&\mathcal{R}^-(0_q, Y, -Z') \\
&= (0_q, \iota Z' - R^-(q)X + R^-(q)\zeta(\cdot)Z, Z) \quad (\because\ \Pi_{\mathfrak{h}} Y_{-1} = Z) \\
&= (0_q, \iota Z' - R^+(q)X + X + R^-(q)\zeta(\cdot)Z, Z) \\
&= (0_q, X, 0) + (0_q, -\zeta(\cdot)Z, Z) \quad (\because\ \iota Z' - R^+(q)X = 0 \textit{ and } R^-(q)\zeta(\cdot)Z = -\zeta(\cdot)Z)
\end{aligned}$$

and this establishes the reverse inclusion $\mathcal{I}^- \oplus \mathcal{Q} \subset Im\mathcal{R}^-$. The assertion that $\mathcal{I}^-$ coincides with the adjoint bundle of $Im\mathcal{R}^-$ is now clear. □

From Proposition 5.1.8, it follows that

$$\begin{aligned}
Im\mathcal{R}^+/\mathcal{I}^+ &= \bigcup_{q\in U} \{0_q\} \times (L^+\mathfrak{g}/L^+\mathfrak{g}) \times \mathfrak{h} \\
&\simeq \bigcup_{q\in U} \{0_q\} \times \{0\} \times \mathfrak{h}
\end{aligned} \tag{5.1.15}$$

where the identification map is given by

$$(0_q, X + L^+\mathfrak{g}, Z) \mapsto (0_q, 0, Z). \tag{5.1.16}$$

Similarly, as a consequence of Proposition 5.1.9, we obtain

$$Im\mathcal{R}^-/\mathcal{I}^- \simeq \mathcal{Q}. \tag{5.1.17}$$

This time, the identification is given by the map

$$(0_q, X, Z) + \mathcal{I}^-_q \mapsto (0_q, -\zeta(\cdot)Z, Z). \tag{5.1.18}$$

The following proposition is now obvious.

Proposition 5.1.10. *The isomorphism $\theta : Im\mathcal{R}^+/\mathcal{I}^+ \longrightarrow Im\mathcal{R}^-/\mathcal{I}^-$ defined in Proposition 2.1.2 (b) is given by*

$$\theta(0_q, 0, Z) = (0_q, -\zeta(\cdot)Z, Z).$$

5.2. Solution of the integrable elliptic spin Calogero-Moser systems.

We are now ready to discuss the factorization problem

$$exp\{\, t(0,0,M(q^0,p^0,\xi^0))\}(q^0) = \gamma_+(t)\,\gamma_-(t)^{-1} \tag{5.2.1}$$

where $(\gamma_+(t),\gamma_-(t)) = ((q^0,k_+(t),q(t)),(q^0,k_-(t),q(t))) \in Im(\mathcal{R}^+,\mathcal{R}^-)$ is to be determined subject to the constraint in (2.2.4) with $\mathfrak{g}$ replaced by $L\mathfrak{g}$ and where $(q^0,p^0,\xi^0) \in J^{-1}(0) = TU \times (\mathcal{U}\cap\mathfrak{h}^\perp)$ is the initial value of (q,p,ξ). (Recall that $M(q^0,p^0,\xi^0)(z) = L(q^0,p^0,\xi^0)(z)/z$.) Note that by the global version of Proposition 5.1.9, we have the unique factorization

$$(q^0,k_-(t),q(t)) = (q^0,\widehat{k}_-(t),q^0)(q^0,e^{\zeta(\cdot)(q^0-q(t))},q(t)) \tag{5.2.2}$$

where $(q^0,\widehat{k}_-(t),q^0)$ is in the Lie group bundle integrating $\mathcal{I}^-$ and the second factor $(q^0,e^{\zeta(\cdot)(q^0-q(t))},q(t))$ is in the Lie groupoid integrating $\mathcal{Q}$. As before, we denote $k_\pm(t)(z)$ by $k_\pm(z,t)$. Also, denote $\widehat{k}_-(t)(z)$ by $\widehat{k}_-(z,t)$. Then $k_+(\cdot,t) \in L^+G$, while $\widehat{k}_-(\cdot,t) = k_-(\cdot,t)e^{\zeta(\cdot)(q(t)-q^0)}$ enjoys the following properties:

(a) $\widehat{k}_-(\cdot,t)$ is holomorphic in $\mathbb{C}\backslash\Lambda$ with singularities at the points of Λ,
(b) $\widehat{k}_-(z+2\omega_i,t) = e^{2\eta_i q^0}\widehat{k}_-(z,t)e^{-2\eta_i q^0}, \quad i=1,2,$
(c) $\left(\frac{d}{dt}\Big|_{t=0}\widehat{k}_-(z,t)\right)_{-1} \in \mathfrak{h}^\perp.$

From the factorization problem on the Lie groupoid above and (5.2.2), it follows that

$$e^{tM(q^0,p^0,\xi^0)(z)} = k_+(z,t)e^{\zeta(z)(q(t)-q^0)}\widehat{k}_-(z,t)^{-1} \tag{5.2.3}$$

where $q(t)$, $k_+(\cdot,t)$ and $\widehat{k}_-(\cdot,t)$ are to be determined. To do so, let us introduce the following gauge transformations of $L(q,p,\xi)$ and $M(q,p,\xi)$:

$$\begin{aligned} L^e(q,p,\xi)(z) &:= Ad_{e^{-\zeta(z)q}}L(q,p,\xi)(z),\\ M^e(q,p,\xi)(z) &:= Ad_{e^{-\zeta(z)q}}M(q,p,\xi)(z) \end{aligned} \tag{5.2.4}$$

for $(q,p,\xi) \in J^{-1}(0) = TU \times (\mathcal{U}\cap\mathfrak{h}^\perp)$. Then the problem in (5.2.3) can be reformulated in the form

$$e^{tM^e(q^0,p^0,\xi^0)(z)} = k_+^s(z,t)k_-^e(z,t)^{-1} \tag{5.2.5}$$

where

$$k_+^s(z,t) = e^{-\zeta(z)q^0}k_+(z,t)e^{\zeta(z)q(t)} \tag{5.2.6}$$

and

$$k_-^e(z,t) = e^{-\zeta(z)q^0}\widehat{k}_-(z,t)e^{\zeta(z)q^0}. \tag{5.2.7}$$

Now, on using the property of $\widehat{k}_-(z,t)$ in property (b) above and the fact that $l(w,z+2\omega_i) = e^{2\eta_i w} l(w,z)$, it is straight forward to check that

$$k^e_-(z+2\omega_i,t) = k^e_-(z,t), \tag{5.2.8}$$

and

$$L^e(q,p,\xi)(z+2\omega_i) = L^e(q,p,\xi)(z) \tag{5.2.9}$$

for $i=1,2$. In view of this, it is natural to introduce the elliptic curve $\Sigma = \mathbb{C}/\Lambda$ where Λ is the rank 2 lattice in (5.1.9). Thus we can regard $k^e_-(\cdot,t)$ as a holomorphic map on $\Sigma\backslash\{0\}$ taking values in G. On the other hand, the factor $k^s_+(\cdot,t)$ in (5.2.6) is holomorphic in a deleted neighborhood of $0 \in \Sigma$. Hence we can think of (5.2.5) as a factorization problem on a small circular contour centered at $0 \in \Sigma$ where $k^s_+(\cdot,t)$ and $k^e_-(\cdot,t)$ have analyticity properties as indicated above and satisfying additional constraints. Indeed, it follows from (5.2.6) and property (c) above for $\widehat{k}_-(\cdot,t)$ that

$$k^s_+(z,t) \sim e^{\frac{-q^0}{z}} k_+(0,t) e^{\frac{q^0}{z}} \quad \text{in a deleted neighborhood of} \quad 0, \tag{5.2.10}$$

$$\left(\frac{d}{dt}\Big|_{t=0} \widehat{k}^e_-(z,t) \right)_{-1} \in \mathfrak{h}^\perp. \tag{5.2.11}$$

Note that if $(q(t),p(t),\xi(t))$ is the solution of the Hamiltonian equations in (5.1.6) satisfying the initial condition $(q(0),p(0),\xi(0)) = (q^0,p^0,\xi^0)$, then by Theorem 2.2.2 and our discussion above, we have

$$\begin{aligned} & L^e(q(t),p(t),\xi(t))(z) \\ = & k^s_+(z,t)^{-1} L^e(q^0,p^0,\xi^0)(z) k^s_+(z,t) \\ = & k^e_-(z,t)^{-1} L^e(q^0,p^0,\xi^0)(z) k^e_-(z,t). \end{aligned} \tag{5.2.12}$$

In the following, we shall write down the solution of the factorization problem explicitly in terms of Riemann theta functions for the case where $\mathfrak{g} = sl(N,\mathbb{C})$ with $\mathfrak{h}$ taken to be the Cartan subalgebra consisting of diagonal matrices in $\mathfrak{g}$. As similar procedures can also be carried out for other classical simple Lie algebras, we shall not give details here for the other cases.

For our purpose, we introduce the spectral curve C as defined by the equation

$$det(L(q^0,p^0,\xi^0)(z) - wI) = 0. \tag{5.2.13}$$

By (5.2.9), this defines an N-sheeted branched covering $\pi : C \longrightarrow \Sigma$ of the elliptic curve Σ. Let

$$I(q^0,p^0,\xi^0;z,w) := det(L(q^0,p^0,\xi^0)(z) - wI) \tag{5.2.14}$$

for $(z,w)\in\mathbb{C}^*\times\mathbb{C}$. We shall make the following genericity assumptions:

(GA1) zero is a regular value of $I(q^0,p^0,\xi^0;\cdot,\cdot)$,

(GA2) the eigenvalues $\lambda_1,\dots,\lambda_N$ of ξ^0 are distinct.

Then the curve C is smooth. The points on C corresponding to $z=0$ will be considered as points "at ∞", we shall denote them by $P_1,\dots,P_N$ respectively. Note that for P on the finite part of C, we have $dim\, ker(L(z(P))-w(P)I)=1$, for otherwise we would obtain a contradiction to assumption (GA1). Consequently, there exists a unique eigenvector $\widehat{v}(P)$ of the matrix $L^e(q^0,p^0,\xi^0)(z(P))$ corresponding to the eigenvalue $w(P)$ normalized by the condition $\widehat{v}_1(P)=(e_1,\widehat{v}(P))=1$. The next result gives a summary on the properties of the spectral curve and $\widehat{v}(P)$ and can be obtained by following the analysis in [KBBT].

Proposition 5.2.1. *Under the genericity assumptions (GA1) and (GA2), the spectral curve C has the following properties:*

(a) C is smooth and is an N-sheeted branched cover of the elliptic curve Σ,

(b) in a deleted neighborhood of $z=0$, C can be represented as

$$\prod_{r=1}^{N}\left(\frac{\lambda_r}{z}+h_r(z)-w\right)=0$$

where $h_1,\dots,h_N$ are holomorphic in a neighborhood of $z=0$,

(c) the genus of C is $g=\frac{1}{2}(N^2-N+2)$.

On the other hand, the components $\widehat{v}_j(P)=(e_j,\widehat{v}(P))$ of the eigenvector $\widehat{v}(P)$, $j=2,\dots,N$, are meromorphic on the finite part of C with polar divisor $D=\sum_{i=1}^{g-1}\gamma_i$. Moreover, in a deleted neighborhood of P_k, we have

$$\widehat{v}_j(P)=e^{-\zeta(z(P))(q_j^0-q_1^0)}(\psi_j^{(k)}+O(z(P))) \tag{5.2.15}$$

where $\psi^{(k)}$ is the eigenvector of ξ^0 corresponding to λ_k with $\psi_1^{(k)}=1$, $k=1,\dots,N$.

Now, from the definition of $\widehat{v}(P)$ and $M^e(q^0,p^0,\xi^0)$, we have

$$M^e(q^0,p^0,\xi^0)(z(P))\,\widehat{v}(P)=(w(P)/z(P))\,\widehat{v}(P). \tag{5.2.16}$$

Hence it follows from (5.2.5) and (5.2.16) that

$$e^{t(w(P)/z(P))}(k_+^s(z(P),t)^{-1}\widehat{v}(P))=k_-^e(z(P),t)^{-1}\widehat{v}(P) \tag{5.2.17}$$

for $z(P)$ in a deleted neighborhood of $0\in\Sigma$. Set

$$v_+(t,P)=k_+^s(z(P),t)^{-1}\widehat{v}(P), \tag{5.2.18}$$

$$v_-(t,P) = k_-^e(z(P),t)^{-1}\widehat{v}(P). \tag{5.2.19}$$

Then from (5.2.17), (5.2.12) and the definition of $\widehat{v}(P)$, we obtain

$$e^{t(w(P)/z(P))}v_+(t,P) = v_-(t,P), \tag{5.2.20}$$

$$L^e(q(t),p(t),\xi(t))(z(P))v_\pm(t,P) = w(P)v_\pm(t,P). \tag{5.2.21}$$

In this way, we are led to scalar factorization problems for the components of a suitably normalized eigenvector of $L^e(q(t),p(t),\xi(t))(z(P))$.

Proposition 5.2.2. *In a deleted neighborhood of P_k, $k=1,\dots,N$, we have*

$$\begin{aligned} v_+^j(t,P) &= e^{\zeta(z(P))(q_1^0-q_j(t))}((k_+(0,t)^{-1}\psi^{(k)})_j + O(z(P))) \\ &\sim e^{(q_1^0-q_j(t))z(P)^{-1}}(k_+(0,t)^{-1}\psi^{(k)})_j \quad as \quad P\to P_k, \end{aligned}$$

$$\begin{aligned} v_-^j(t,P) &= e^{tw(P)z(P)^{-1}+\zeta(z(P))(q_1^0-q_j(t))}((k_+(0,t)^{-1}\psi^{(k)})_j + O(z(P))) \\ &\sim e^{t(\lambda_k/z(P)+h_k(0))z(P)^{-1}+(q_1^0-q_j(t))z(P)^{-1}}(k_+(0,t)^{-1}\psi^{(k)})_j \quad as \quad P\to P_k. \end{aligned}$$

Proof. This is a consequence of (5.2.18)-(5.2.20),(5.2.6),(5.2.15) and the fact that in a deleted neighborhood of P_k, we have $w(P) = \lambda_k z(P)^{-1} + h_k(0) + O(z(P))$ from Proposition 5.2.1 (b). □

In order to write down $v_-^j(t,P)$, we will insert a fictitious pole together with a matching zero to this function at some point γ_0 on the finite part of C distinct from $\gamma_1,\dots,\gamma_{g-1}$. By putting in an additional pole in this way, we would be able to construct $v_-^j(t,P)$ as a multi-point Baker-Akheizer function. To do so, let us fix a canonical homology basis $\{a_j,b_k\}_{1\le j,k\le g}$ of the Riemann surface associated with C and let $\{\omega_i\}_{1\le i\le g}$ be a cohomology basis dual to $\{a_j,b_k\}_{1\le j,k\le g}$, i.e. $\int_{a_j}\omega_i = \delta_{ij}$, $\int_{b_j}\omega_i = \Omega_{ij}$. With respect to the Riemann matrix $\Omega=(\Omega_{ij})$, we construct the theta function

$$\theta(z_1,\dots,z_g) = \sum_{m\in\mathbb{Z}^g} exp\{2\pi i(m,z) + \pi i(\Omega m,m)\}. \tag{5.2.22}$$

We also introduce the Abel-Jacobi map

$$A: C \longrightarrow Jac(C), \quad P \mapsto \left(\int_{P_0}^P \omega_1,\dots,\int_{P_0}^P \omega_g\right) \tag{5.2.23}$$

where P_0 is some fixed point on the finite part of C. Now, let $d\Omega^{(i)}$, $i = 1, 2$, be the unique abelian differential of second kind with vanishing a-periods such that in a deleted neighborhood of P_k,

$$d\Omega^{(1)} = d(z^{-1} + \omega^{(1)}(z)), \tag{5.2.24}$$

$$d\Omega^{(2)} = d(\lambda_k z^{-2} + h_k(0)z^{-1} + \omega^{(2)}(z)), \tag{5.2.25}$$

where $\omega^{(1)}(z), \omega^{(2)}(z)$ are regular at $z = 0$, $k = 1, \ldots, N$. We shall denote by $2\pi i U^{(i)}$ the vector of b-periods of $d\Omega^{(i)}$, $i = 1, 2$. Then from Proposition 5.2.2 and the fact that $v_-^j(t, P)$ has $\sum_{i=0}^{g-1} \gamma_i$ as a polar divisor, we obtain the following result from the standard construction of Baker-Akhiezer functions [K].

Proposition 5.2.3. *For* $1 \le j \le N$,

$$\begin{aligned} &v_-^j(t, P) \\ &= f_j(t) \frac{\theta(A(P) + (q_1^0 - q_j(t))U^{(1)} + tU^{(2)} - A(D) - A(\gamma_0) - K)}{\theta(A(P) - A(D) - A(\gamma_0) - K)} \\ &\quad \times exp\,[(q_1^0 - q_j(t))\Omega^{(1)}(P) + t\,\Omega^{(2)}(P)] \end{aligned}$$

where

$$\Omega^{(i)}(P) = \int_{P_0}^{P} d\,\Omega^{(i)}, \quad i = 1, 2$$

and K *is the vector of Riemann constants.*

Corollary 5.2.4. $\theta\left(q_j(t)U^{(1)} - tU^{(2)} + V\right) = 0$ *where* $V = A(D) - q_1^0 U^{(1)} + K$, $j = 1, \ldots, N$.

Proof. This follows when we evaluate the expression for $v_-^j(t, P)$ in Proposition 5.2.3 at the point γ_0 and equate the result to zero. $\square$

Let

$$f(t) = diag(f_1(t), \ldots, f_N(t)). \tag{5.2.26}$$

In view of Proposition 5.2.3, we shall write

$$v_-(t, P) = f(t)\, v_-^\theta(t, P) \tag{5.2.27}$$

where the $v_-^\theta(t, P)$ are known. Note that if we set $t = 0$ in the above expression, we obtain $\widehat{v}(P) = f(0)\, v_-^\theta(0, P)$. Clearly, $f_1(0) = 1$; the other $f_j(0)$'s are then uniquely determined from the definition of $\widehat{v}(P)$. Now, for given $z \in \Sigma$ which is not a branch

point of the coordinate function $z(P)$, there exist N points $P_1(z), \dots, P_N(z)$ of C lying over z. Hence we can define the matrices

$$\widehat{V}(z) = (\widehat{v}(P_1(z)), \dots, \widehat{v}(P_N(z))), \quad V_-(z,t) = (v_-(t, P_1(z)), \dots, v_-(t, P_N)),$$
$$V_-^{\theta}(z,t) = (v_-^{\theta}(t, P_1(z)), \dots, v_-^{\theta}(t, P_N(z))). \tag{5.2.28}$$

With these definitions, if follows from (5.2.27) and (5.2.19) that

$$k_-^e(z,t) = \widehat{V}(z) V_-^{\theta}(z,t)^{-1} f(t)^{-1} \tag{5.2.29}$$

where $f(t)$ is still to be determined. To do this, we invoke the condition that $T_{\gamma_+(t)} \boldsymbol{l}_{\gamma_+(t)^{-1}} \dot{\gamma}_+(t) \in \mathcal{R}^+(\{q(t)\} \times \{0\} \times L\mathfrak{g})$. Indeed, it follows from the proof of Theorem 2.2.2 in [L2] that

$$T_{\gamma_+(t)} \boldsymbol{l}_{\gamma_+(t)^{-1}} \dot{\gamma}_+(t) = \mathcal{R}^+(q(t), 0, M(q(t), p(t), \xi(t))). \tag{5.2.30}$$

Consequently, we have

$$\begin{aligned} &k_+(z,t)^{-1} \dot{k}_+(z,t) \\ =\, &(R^+(q(t)) M(q(t), p(t), \xi(t)))(z) \\ =\, &M(q(t), p(t), \xi(t))(z) - \zeta(z) p(t) + \sum_{i \neq j} l(q_i(t) - q_j(t), z)(\zeta(q_i(t) - q_j(t)) \\ &+ \zeta(z) - \zeta(q_i(t) - q_j(t) + z)) \xi_{ij}(t) e_{ij} \end{aligned} \tag{5.2.31}$$

where in the last step we have used (5.1.8). Therefore, when we expand the above expression about $z = 0$ and compare the term in z^0, we find that

$$k_+(0,t)^{-1} \dot{k}_+(0,t) = \sum_{i \neq j} \zeta'(q_i(t) - q_j(t)) \xi_{ij}(t) e_{ij} \tag{5.2.32}$$

from which we obtain the condition

$$\Pi_{\mathfrak{h}} \Big((k_+(0,t)^{-1} \dot{k}_+(0,t) \Big) = 0. \tag{5.2.33}$$

In order to state our next result, we introduce

$$\begin{aligned} \omega_k^{(1)} &= \lim_{P \to P_k} \Big(\Omega^{(1)}(P) - z(P)^{-1} \Big) \\ \omega_k^{(2)} &= \lim_{P \to P_k} \Big(\Omega^{(2)}(P) - (\lambda_k z(P)^{-2} + h_k(0) z(P)^{-1}) \Big) \end{aligned} \tag{5.2.34}$$

for $k = 1, \dots, N$. We also introduce the matrix $W^{\theta}(t) = (W_{jk}^{\theta}(t))$ where

$$\begin{aligned} W_{jk}^{\theta}(t) = \,&\frac{\theta(A(P_k) + (q_1^0 - q_j(t)) U^{(1)} + t U^{(2)} - A(D) - A(\gamma_0) - K)}{\theta(A(P_k) - A(D) - A(\gamma_0) - K)} \\ &\times exp \left[(q_1^0 - q_j(t)) \omega_k^{(1)} + t \omega_k^{(2)} \right]. \end{aligned} \tag{5.2.35}$$

Proposition 5.2.5. *$f(t)$ satisfies the differential equation*

$$\dot{f}(t) = -f(t)\,\Pi_{\mathfrak{h}}\left(\dot{W}^{\theta}(t)W^{\theta}(t)^{-1}\right)$$

and hence

$$f(t) = f(0)\,exp\left\{-\int_0^t \Pi_{\mathfrak{h}}\left(\dot{W}^{\theta}(\tau)W^{\theta}(\tau)^{-1}\right)\,d\tau\right\}. \tag{5.2.36}$$

Proof. From Proposition 5.2.2 and 5.2.3, we have

$$\begin{aligned}
&(k_+(0,t)^{-1}\psi^{(k)})_j \\
&= \lim_{P\to P_k} v_-^j(t,P)exp\left[-t(\lambda_k z(P)^{-2} + h_k(0)z(P)^{-1}) - (q_1^0 - q_j(t))z(P)^{-1}\right] \\
&= f_j(t)W^{\theta}_{jk}(t)
\end{aligned}$$

which implies

$$k_+(0,t) = \Psi W^{\theta}(t)^{-1}f(t)^{-1}$$

where Ψ is the $N\times N$ matrix whose k-th column is the vector $\psi^{(k)}$, $k = 1,\dots,N$. Differentiating the above expression with respect to t, we find

$$k_+(0,t)^{-1}\dot{k}_+(0,t) = -f(t)\dot{W}^{\theta}W^{\theta}(t)^{-1}f(t)^{-1} - \dot{f}(t)f(t)^{-1}.$$

Therefore, when we apply the condition in (5.2.33), the desired equation for $f(t)$ follows. Finally, the solution of the equation is obvious. □

Hence $k_+(0,t) = \Psi W^{\theta}(t)^{-1}f(t)^{-1}$ and $k_-^e(z,t) = \widehat{V}(z)V_-^{\theta}(z,t)^{-1}f(t)^{-1}$ are completely determined. Therefore we have following result.

Theorem 5.2.6. *Let $(q^0,p^0,\xi^0) \in J^{-1}(0) = TU\times(\mathcal{U}\cap\mathfrak{h}^{\perp})$ satisfy the genericity assumptions (GA1), (GA2). Then the Hamiltonian flow on $J^{-1}(0)$ generated by*

$$\mathcal{H}(q,p,\xi) = \frac{1}{2}\sum_i p_i^2 - \frac{1}{2}\sum_{i\neq j}\wp(q_i-q_j)\xi_{ij}\xi_{ji}$$

with initial condition $(q(0),p(0),\xi(0)) = (q^0,p^0,\xi^0)$ is given by

$$\begin{aligned}
&\theta\left(q_j(t)U^{(1)} - tU^{(2)} + V\right) = 0, \quad j = 1,\dots,N, \\
&\xi(t) = k_+(0,t)^{-1}\xi^0\,k_+(0,t), \\
&p(t) = Ad_{k_-^e(z,t)^{-1}}L^e(q^0,p^0,\xi^0)(z) \\
&\qquad + \sum_{i\neq j} l(q_i(t)-q_j(t),z)e^{-\zeta(z)(q_i(t)-q_j(t))}\xi_{ij}(t)e_{ij}
\end{aligned} \tag{5.2.37}$$

where $k_+(0,t)$, $k_-^e(z,t)$ are given by the formulas above.

Finally we are ready to give the solutions of the associated integrable model on $TU\times\mathfrak{g}_{red}$ whose equations are given in Proposition 5.1.2.

Corollary 5.2.7. *Let* $(q^0, p^0, s^0) \in TU \times \mathfrak{g}_{red}$ *and suppose* $s^0 = Ad_{g(\xi^0)^{-1}}\xi^0$ *where* $\xi^0 \in \mathcal{U} \cap \mathfrak{h}^\perp$. *Then the Hamiltonian flow generated by* $\mathcal{H}_0$ *with initial condition* $(q(0), p(0), s(0)) = (q^0, p^0, s^0)$ *is given by*

$$
\begin{aligned}
&\theta\left(q_j(t)U^{(1)} - tU^{(2)} + V\right) = 0, \quad j = 1, \dots, N, \\
&s(t) = Ad_{\left(\widetilde{k}_+(0,t)\, g\left(Ad_{\widetilde{k}_+(0,t)^{-1}}\, s^o\right)\right)^{-1}} s^0, \\
&p(t) = Ad_{\left(\widetilde{k}^e_+(z,t)\, g\left(Ad_{\widetilde{k}_+(0,t)^{-1}}\, s^0\right)\right)^{-1}} L^e(q^0, p^0, s^0)(z) \\
&\qquad + \sum_{i \neq j} l(q_i(t) - q_j(t), z) e^{-\zeta(z)(q_i(t) - q_j(t))} s_{ij}(t) e_{ij}
\end{aligned}
\tag{5.2.38}
$$

where $k_+(0,t)$, $k^e_-(z,t)$ *are given by the formulas above.*

Remark 5.2.8. (a) In [KBBT], the authors considered the $gl(N, \mathbb{C})$-elliptic spin Calogero-Moser system with Hamiltonian

$$H(q, p, f) = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i \neq j} \wp(q_i - q_j)\xi_{ij}\xi_{ji} \tag{5.2.39}$$

and they imposed the following restriction on $\xi = (\xi_{ij}) \in gl(N, \mathbb{C}) \simeq gl(N, \mathbb{C})^*$, namely, they set

$$\xi_{ij} = b_i^T a_j \tag{5.2.40}$$

for all i and j where a_j, b_j are (column) vectors in $\mathbb{C}^l$, $l < N$, satisfying the nontrivial Poisson bracket relations $\{a_{i,\alpha}, b_{j,\beta}\} = \delta_{i,j}\delta_{\alpha,\beta}$. Thus from the outset, it is clear that these authors were restricting themselves to special coadjoint orbits of $gl(N, \mathbb{C})^* \simeq gl(N, \mathbb{C})$ which consist of matrices of the form $B^T A$, where

$$A = (a_1, \cdots, a_N), \;\; B = (b_1, \cdots, b_N) \tag{5.2.41}$$

are $l \times N$ matrices. However, it is only through the imposition of (5.2.40) that they were able to make the connection with the matrix KP equation. The precise relation is that the equations of motion for a_j, b_j (up to gauge equivalence) and q_j are the necessary and sufficient condition for the time-dependent matrix Schrödinger equation

$$\left(\partial_t - \partial_x^2 + \sum_{j=1}^{N} a_j(t) b_j^T(t) \wp(x - q_j(t))\right)\Psi = 0 \tag{5.2.42}$$

and its adjoint

$$\widetilde{\Psi}^T\left(\partial_t - \partial_x^2 + \sum_{j=1}^{N} a_j(t) b_j^T(t) \wp(x - q_j(t))\right) = 0 \tag{5.2.43}$$

$(\widetilde{\Psi}^T\partial \equiv -\partial\widetilde{\Psi}^T)$ to admit solutions of the form

$$\Psi = \sum_{j=1}^{N} s_j(t,k,z)\Phi(x-q_j(t),z)e^{kx+k^2t}, \tag{5.2.44}$$

$$\widetilde{\Psi} = \sum_{j=1}^{N} s_j^+(t,k,z)\Phi(-x+q_j(t),z)e^{-kx-k^2t}, \tag{5.2.45}$$

where s_j,s_j^+ are functions which take values in $\mathbb{C}^l$ and

$$\Phi(x,z) = \frac{\sigma(z-x)}{\sigma(x)\sigma(z)}e^{\zeta(z)\,x}. \tag{5.2.46}$$

Furthermore, it is in the course of proving this result that the Lax pair as well as the constraint $f_{jj} = b_j^T a_j = 2, j = 1,\cdots,N$, emerge naturally. In short, the method of solution in [KBBT] of their elliptic spin Calogero-Moser system as defined in (5.2.39), (5.2.40) on the constraint manifold $\{f_{jj} = b_j^T a_j = 2, j = 1,\cdots,N\}$ is based on the above correspondence.

In what follows, we shall give a sketch of this method to solve the equations of motion for $a_j(t), b_j(t)$ and $q_j(t)$ so that the reader can understand its limitations. To start with, one has a normalized Baker-Akhiezer vector function $\psi(x,t,P)$ (and its dual $\psi^+(x,t,P)$) which is uniquely determined by the spectral curve, a divisor of degree $g+l-1$, and prescribed behaviour in deleted neighborhoods of the punctures P_j, $j = 1,\cdots,l$. (These are in turn fixed by the initial data.) From a general result in [K], corresponding to $\psi(x,t,P)$ and $\psi^+(x,t,P)$ is an algebro-geometric potential $u(x,t)$ satisfying

$$\left(\partial_t - \partial_x^2 + u(x,t)\right)\psi(x,t,P) = 0 \tag{5.2.47}$$

and

$$(\psi^+(x,t,P))^T\left(\partial_t - \partial_x^2 + u(x,t)\right) = 0. \tag{5.2.48}$$

Next, by analyzing $\psi(x,t,P)$ as a function of x, and on comparing $\Psi(x,t,P)$ with $\psi(x,t,P)$, one can show that there exists a constant invertible matrix χ_0 such that

$$\chi_0 u(x,t)\chi_0^{-1} = \sum_{j=1}^{N} a_j(t)b_j^T(t)\wp(x-q_j(t)). \tag{5.2.49}$$

Since $u(x,t)$ is determined by $\psi(x,t,P)$, it is in this fashion that the authors in [KBBT] were able to write down $a_j(t), b_j(t)$ and the equation satisfied by $q_j(t)$ in terms of theta functions. Now, let us examine the expression $\sum_{j=1}^{N} a_j(t)b_j^T(t)\wp(x-$

$q_j(t))$ for the potential carefully. Clearly, it depends on the a_j's and b_j's through the rank one matrices $a_j b_j^T$ rather than on the entries of the matrix $\xi = (b_j^T a_j)$. For this reason, the solution method sketched above is rather specific to these special coadjoint orbits of $gl(N, \mathbb{C})^*$. Obviously, similar remarks also hold for the corresponding rational and trigonometric cases. Thus it is clear that this method is not applicable to our more general class of spin Calogero-Moser systems associated with simple Lie algebras here.

(b) The first link between elliptic solutions of integrable PDEs and discrete particle systems was found in the paper of Airault, McKean and Moser [AMM]. Indeed, the PDE in [AMM] is KdV and the corresponding discrete particle system is the usual (spinless) Calogero-Moser system. In the context of the spin CM system defined by (5.2.39) and (5.2.40) above, we remark that its correspondence with matrix KP is related to some interesting algebraic geometry and we refer the reader to [T] for details. For our general class of spin CM systems which we address in this work, whether it has any connection with integrable PDEs is entirely open at this point.

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